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Tutorial 2:
THE FIRST STEP TO THE DIRECT INTEGRAL FORMS

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Summary and objectives

In this tutorial we will introduce the reader to how the direct integral forms for a partial differential equation can be derived. We will start from the well-known form for the integration by parts rule and then we will generalize it to demonstrate how Green's second identity works. Therefore the main objectives of this tutorial are:

1. To review the philosophy behind the integration by part rule.
2. To generalize the integration by parts formulae to the form of Green's second identity for multi-dimensional system.
3. To review the rules of indicial notation.
4. To derive the integral equation formulation for Laplace equation.
5. To extend the above formulation to Poisson's equation.
6. To give an overview of the different possible research areas in the BEM.

1 Introduction

In the last tutorial, we describe the difficulties of boundary elements so difficult. We also highlighted the different sources of errors that appear in boundary element codes.
In this tutorial we will show that the basic idea behind boundary elements is the same as the integration by parts rule, which is very well-known and used by most of the readers.

2 Integration by parts

Consider u and v are two functions of the independent variable x in one-dimensional space (see Fig. 1). The following integration by parts formulae is very well known in mathematics:

$$\int_{x=x_1}^{x=x_2} u(x)\, dv(x) = \left[u(x)\, v(x)\right]_{x=x_1}^{x=x_2} - \int_{x=x_1}^{x=x_2} v(x)\, du(x) \tag{1}$$

Equation (1) can be re-written in a more convenient notation as follows:

$$\int_{x=x_1}^{x=x_2} u(x)\, v'(x) dx = \left[u(x)\, v(x)\right]_{x=x_1}^{x=x_2} - \int_{x=x_1}^{x=x_2} v(x)\, u'(x) dx \tag{2}$$

where the (•)' denotes the derivatives of (•) with respect to x.

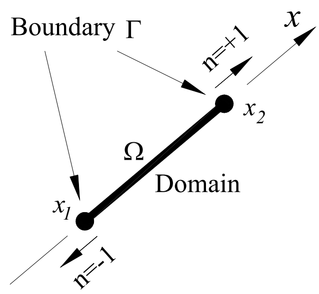

Figure 1: Definitions for the 1D integration by parts.

We shall now examine the former formulae in more depth. Consider the first term:

$$\left[u(x)\, v(x)\right]_{x=x_1}^{x=x_2} = \left[u(x)\, v(x)\right]_{\text{at } x=x_2} - \left[u(x)\, v(x)\right]_{\text{at } x=x_1} \tag{3}$$

This appears strange as the term is a result of an integration process, which always (by definition) involves summation. Now the question is: from where came the minus sign in the last equation? The answer is simple as this term originally is the summation of (u v n) at the boundary points x_1 and x_2 (noting that n is the normal to the problem boundary, see Figure 1). Therefore this term can be re-written as follows:

$$\begin{aligned}\left[u(x)\, v(x)\right]_{x=x_1}^{x=x_2} &= \left[u(x)\, v(x)\, n(x)\right]_{\text{at } x=x_2} + \left[u(x)\, v(x)\, n(x)\right]_{\text{at } x=x_1} \\ &\Rightarrow \sum_{x=x_1,x_2} u(x) v(x) n(x) \Rightarrow \int_\Gamma u(x) v(x) n(x) d\Gamma\end{aligned} \tag{4}$$

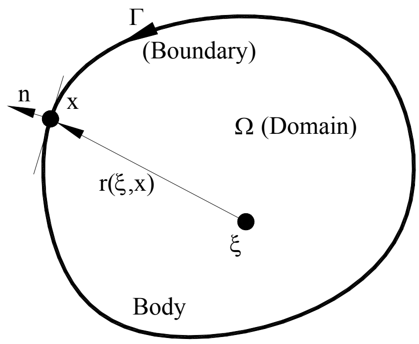

Figure 2: Body definitions.

The second term on the left hand side of equation (2) is an integration over the domain, which can be written in more generalized form as follows:

$$\int_{x=x_1}^{x=x_2} v(x)\, u'(x) dx = \int_{\Omega} v(x) u'(x) d\Omega \tag{5}$$

From the results of equations (4) and (5), equation (2) can be re-written in a more generalized form as follows:

$$\int_{\Omega} u(x)\, v'(x) d\Omega = \int_{\Gamma} u(x)\, v(x) n(x) d\Gamma - \int_{\Omega} v(x)\, u'(x) d\Omega \tag{6}$$

This is Green's second identity for the one-dimensional problems. For higher derivatives, equation (6) can be generalized as follows:

$$\int_{\Omega} u(x)\, v''(x) d\Omega = \int_{\Gamma} u(x)\, v'(x) n(x) d\Gamma - \int_{\Omega} v'(x)\, u'(x) d\Omega \tag{7}$$

Applying the integration by parts of the second integral on the left hand side of equation (7) it gives:

$$\int_{\Omega} v'(x)\, u'(x) d\Omega = \int_{\Gamma} u'(x)\, v(x) n(x) d\Gamma - \int_{\Omega} v(x)\, u''(x) d\Omega \tag{8}$$

Therefore the final formulae can be written as follows:

$$\int_{\Omega} u(x)\, v''(x) d\Omega = \int_{\Gamma} \left[u(x)\, v'(x) n(x) - u'(x)\, v(x) n(x) \right] d\Gamma + \int_{\Omega} v(x)\, u''(x) d\Omega \tag{9}$$

where $\Gamma=(x_1, x_2)$ is the boundary of the domain $\Omega=(x_1 \rightarrow x_2)$. Before we go further it is important to note the following:

1- The main idea of the integration by parts, whether it is in equation (6) or (9), is to swap the differential operator from function *v* to the function *u*.
2- In doing such swapping some boundary terms appear (recall the first term on the right hand side of equation (6) or (9)).
3- In equation (6) the integration by parts is done only once. Therefore the last domain integral on the RHS having a negative sign; whereas when the integration by parts is carried out twice this integral appears with a positive sign as in equation (9). Generally the sign of this integral is equal to $(-1)^m$ where *m* in the number of times the integration by parts is carried out.

Commonly, in the BEM the integration by parts is carried out twice, however in some cases it is carried out only once or as many as four times.

If equations (6) and (9) are generalized to a higher dimensional system (for example 2D or 3D) we obtain the following form (consider Figure 2) [1]:

$$\int_{\Omega} u \,\mathrm{L}v \, d\Omega = \int_{\Gamma} u \,\mathrm{L}^* \; v \, d\Omega + \int_{\Omega} v \,\mathrm{L}^{\mathrm{adj}} u \, d\Omega \tag{10}$$

where L is a deferential operator, $\mathrm{L}^{\mathrm{adj}}$ is the adjoint differential operator (which will be covered in future tutorials), and L* is the differential operator defined by n_α $\mathrm{L}^*=\mathrm{L}_{,\alpha}$. Herein the indicial notation is used and will be discussed in the next section. This equation seems difficult to understand, however, it will be clarified when we illustrate it using the example of the Laplace equation (see section 4).

It is worth noting that originally the integration by parts formulae is derived from Green's second identity. However, in the former explanations we approached the Green's second identity from the integration by parts for the sake of clarity.

3 Indicial (Tensor) notation

In much BEM literature the tensor notation is used (see Ref. [1]). This notation was introduced by Einstein. Herein we will review some basic principles, which will be used from now and henceforth in future tutorials. In the following sub sections we will consider only the case of the 2D formulation; therefore the indexes will vary from 1 to 2. The three-dimensional case can be treated in a similar manner but with indexes varying from 1 to 3.

3.1 The Kronecker delta symbol:

This symbol represents the identity matrix as follows:

$$\begin{aligned} \delta_{\alpha\beta} &= 1 \quad \text{if} \quad \alpha = \beta \\ &= 0 \quad \text{if} \quad \alpha \neq \beta \end{aligned} \tag{11}$$

3.2 The Vector r:

Consider two points ξ and x in the x_i space as shown in Figure 2. The coordinates of the point x, for example, can be referred to as $x_1(\mathrm{x})$, $x_2(\mathrm{x})$ or as in the short tensor notation as $x_\alpha(\mathrm{x})$ where α=1 and 2. The Euclidian distance between two points ξ and x in the x_i space (always positive) is defined using the vector r or $r(\xi,\mathrm{x})$ as follows:

$$r = \sqrt{(x_1(\mathrm{x}) - x_1(\xi))^2 + (x_2(\mathrm{x}) - x_2(\xi))^2} \tag{12}$$

In order to write r in the indicial notation form, we will make use of one rule for the tensor notation states: repeated indexes denote summation, for example: $G_{\theta\theta}=G_{11}+G_{22}$. It has to be noted that in this case the index θ can be replaced by any other symbol without affecting the final results; therefore the index θ is said to be dummy index. It has to be noted that dummy indexes usually appear in pairs and cannot be repeated. Using the summation rule the vector r can be rewritten as:

$$r^2 = (x_\alpha(\mathrm{x}) - x_\alpha(\xi))(x_\alpha(\mathrm{x}) - x_\alpha(\xi)) \tag{13}$$

It is interesting that if the summation rule is applied to the Kronecker delta for 2D problems, it gives (noting that α is a dummy index):

$$\delta_{\alpha\alpha} = \delta_{11} + \delta_{22} = 2 \tag{14}$$

And similarly, for 3D problems: $\delta_{\alpha\alpha}$=3.

3.3 Derivatives of *r*:

The vector *r* is defined from the point ξ (the source point) to the point x (the field point). In the indicial notation, the comma is used to denote derivatives with respect to the coordinates of the field point, as follows:

$$\frac{\partial r}{\partial x_\alpha(\mathrm{x})} = r,_\alpha = -\frac{\partial r}{\partial x_\alpha(\xi)} \tag{15}$$

Using the definition of r in equation (13), and by differentiating both sides with respect to the coordinate of the field point, we can obtain:

$$2r\frac{\partial r}{\partial x_\beta(\mathrm{x})} = (x_\alpha(\mathrm{x}) - x_\alpha(\xi))\frac{\partial(x_\alpha(\mathrm{x}) - x_\alpha(\xi))}{\partial x_\beta(\mathrm{x})} + (x_\alpha(\mathrm{x}) - x_\alpha(\xi))\frac{\partial(x_\alpha(\mathrm{x}) - x_\alpha(\xi))}{\partial x_\beta(\mathrm{x})} \tag{16}$$

or:

$$r\frac{\partial r}{\partial x_\beta(\mathrm{x})} = (x_\alpha(\mathrm{x}) - x_\alpha(\xi))\frac{\partial(x_\alpha(\mathrm{x}) - x_\alpha(\xi))}{\partial x_\beta(\mathrm{x})} \tag{17}$$

Noting that:

$$\frac{\partial(x_\alpha(\mathrm{x}))}{\partial x_\beta(\mathrm{x})} = \delta_{\alpha\beta} \qquad \text{and} \qquad \frac{\partial(x_\alpha(\xi))}{\partial x_\beta(\mathrm{x})} = 0 \tag{18}$$

Then:

$$\begin{aligned} \frac{\partial r}{\partial x_\beta(\mathrm{x})} &= \frac{(x_\alpha(\mathrm{x}) - x_\alpha(\xi))}{r}\delta_{\alpha\beta} \\ &= \frac{(x_\beta(\mathrm{x}) - x_\beta(\xi))}{r} = \frac{r_\beta}{r} = r,_\beta \end{aligned} \tag{19}$$

It has to be noted that only a minus sign is the difference between the derivation with respect to the field and those with respect to the source point (see equation (15)). In most of BEM books the comma denotes the derivatives with respect to the coordinate of the fields point; otherwise it will be explicitly stated.

3.4 Higher derivatives of r:

$$r,_{\alpha\beta} = \frac{\partial r,_\alpha}{\partial x_\beta(\mathrm{x})} = \frac{\partial}{\partial x_\beta(\mathrm{x})}\frac{r_\alpha}{r} = \frac{\partial}{\partial x_\beta(\mathrm{x})}\left[\frac{(x_\alpha(\mathrm{x}) - x_\alpha(\xi))}{r}\right]$$

$$= \frac{r(\delta_{\alpha\beta} - 0) - (x_\alpha(\mathrm{x}) - x_\alpha(\xi))r,_\beta}{r^2}$$

$$= \frac{\delta_{\alpha\beta}}{r} - \frac{r_\alpha r,_\beta}{r^2}$$

$$= \frac{\delta_{\alpha\beta} - r,_\alpha r,_\beta}{r} \tag{20}$$

By default, the former derivatives are taken with respect to the coordinate of the field point. If these derivatives were taken with respect to the coordinate of the source point, it leads to:

$$r,_{\alpha\beta} = \frac{\partial r,_\alpha}{\partial x_\beta(\xi)}$$

$$= \frac{r,_\alpha r,_\beta - \delta_{\alpha\beta}}{r} \tag{21}$$

which lead to the same result as in equation (20), but with minus sign, as previously mentioned. Another higher derivative of r can be obtained in similar way.

3.5 The Laplacian of a function:

A Laplacian of a function F is defined as:

$$\nabla^2 F = \left(\frac{\partial^2}{\partial x_1^2(\mathrm{x})} + \frac{\partial^2}{\partial x_2^2(\mathrm{x})}\right)F \tag{22}$$

or

$$\nabla^2 F = F,_{\alpha\alpha} \tag{23}$$

In this case α is a dummy index. If the Laplacian of a function is equal to zero, the function is said to be a harmonic function.

3.6 The Bi-harmonic operator:

The bi-harmonic operator (∇^4) is defined as follows:

$$\nabla^4 F = \left(\frac{\partial^2}{\partial x_1^2(\mathrm{x})} + \frac{\partial^2}{\partial x_2^2(\mathrm{x})}\right)\left(\frac{\partial^2}{\partial x_1^2(\mathrm{x})} + \frac{\partial^2}{\partial x_2^2(\mathrm{x})}\right)F$$

$$= \left(\frac{\partial^4}{\partial x_1^4(\mathrm{x})} + 2\frac{\partial^4}{\partial x_1^2(\mathrm{x})\partial x_2^2(\mathrm{x})} + \frac{\partial^4}{\partial x_2^4(\mathrm{x})}\right)F \tag{23}$$

or

$$\nabla^4 F = \nabla^2 \nabla^2 F = F_{,\alpha\alpha\beta\beta} \tag{24}$$

Note in the above definition that dummy pairs of indexes cannot be repeated.

4 Laplace Equation

In this section, the integral equation formulation for the Laplace equation will be derived step by step. The Laplace equation governs many problems in engineering and physics, such as torsion of solids and potential flow. The differential form for this equation is given as follows:

$$\nabla^2 u(\mathrm{x}) = 0 \tag{25}$$

This governing equation is defined for a certain problem, for example the problem shown in Figure 2. The density $u(\mathrm{x})$ is the potential at any point x inside the domain or on the boundary. The notation $u(\mathrm{x})$ denotes that u is function of the coordinate of the point x; therefore it is an abbreviation for the notation $u(x_1(\mathrm{x}), x_2(\mathrm{x}))$. In order to establish the direct integral form for differential equation (25), the following *integral identity* is considered:

$$\int_{\Omega} \left(\nabla^2 u(\mathrm{x})\right) U^* \, d\Omega = 0 \tag{26}$$

where U^* is a weighing function or functional. It has to be noted that, if this weighting function is chosen to be an approximate function to minimize the error in the solution of u, the former statement is said to be a *weighted residual statement* as in the case of the finite element method. In the case of the BEM, as we will discuss later, we do not need a weighted residual statement as the U^* will be chosen as analytical and exact kernels.

In the next steps of derivation we will follow the same steps as from equation (7) by replacing v'' by the $\nabla^2 u(\mathrm{x})$ and $u(\mathrm{x})$ by U^* to obtain a similar equation to equation (9). In order to do so, we apply Green's Identity (or the integration by parts) to equation (26). It gives:

$$\int_{\Gamma} u_{,\alpha}(\mathrm{x}) n_{\alpha}(\mathrm{x}) U^* \, d\Gamma - \int_{\Omega} u_{,\alpha}(\mathrm{x}) U^*_{,\alpha} \, d\Omega = 0 \tag{27}$$

Again, applying Green's identity to the second integral in equation (27), gives:

$$\int_{\Omega} u_{,\alpha}(\mathrm{x}) U^*_{,\alpha} \, d\Omega = \int_{\Gamma} u(\mathrm{x}) \, n_{\alpha}(\mathrm{x}) U^*_{,\alpha} d\Gamma - \int_{\Omega} u(\mathrm{x}) U^*_{,\alpha\alpha} \, d\Omega \tag{28}$$

Noting that:

$$q = \frac{\partial u}{\partial n} = u_{,\alpha} \, n_{\alpha} \qquad \text{and} \qquad q^* = U^*_{,\alpha} \, n_{\alpha} \tag{29}$$

where q is the flux. Equation (28) can be re-written in the following form:

$$\int_{\Gamma} q(\mathrm{x}) U^* \, d\Gamma - \int_{\Gamma} u(\mathrm{x}) \, q^* d\Gamma + \int_{\Omega} u(\mathrm{x}) \nabla^2 U^* \, d\Omega = 0 \tag{30}$$

It can be seen from the last equation that the first two integrals are boundary terms (a similar result to that of equation (9)) and the last integral is a domain integral term which contains the adjoint operator (which is the Laplacian). The last integral represents the *first type* of domain integrals that appear in the BEM. It is important, now, to show how to get rid of this domain integral.

Before considering the domain integral, we can show how U^* can be selected:
Let u(x)=c (constant) then q(x)=0 and:

$$\int_\Omega \nabla^2 U^* \, d\Omega = \int_\Gamma q^* d\Gamma \tag{31}$$

which, by definition, is a trivial solution (satisfying Green's second identity). However the identity in equation (31) is very useful and we will make extensive use of it in future tutorials on transforming domain integrals to the boundary.

Let $U^*(\mathrm{x})=c$ then $q^*(\mathrm{x})=0$ and:

$$\int_\Gamma q(\mathrm{x}) \, d\Gamma = 0 \tag{32}$$

which means the integration of the flux along a closed boundary is zero, which matches the rules of physics (flux equilibrium). In elasticity problem q will denote tractions and in this case the above equation represents the equilibrium.

Recalling equation (30), In order to get rid of the last domain integral the following case could be used:

$$\nabla^2 U^*(\xi, \mathrm{x}) = -\delta(\xi, \mathrm{x}) \tag{33}$$

where $\delta(\xi,\mathrm{x})$ is the Paul Dirac delta which is not a function, it is a distribution or a functional. The Dirac delta is defined as zero everywhere, except at $\xi=\mathrm{x}$ where it is infinity. The singular particular solution of equation (33) is called the fundamental solution. Now the $U^*(\xi,\mathrm{x})$ are called two-point kernels. Substituting equation (33) into the last domain integral in equation (30) gives:

$$\int_\Omega u(\mathrm{x}) \, \nabla^2 U^*(\xi, \mathrm{x}) \, d\Omega = -\int_\Omega u(\mathrm{x}) \, \delta(\xi, \mathrm{x}) \, d\Omega = -u(\xi) \tag{34}$$

where equation (34) is a well-know property of the Dirac delta distribution. It can be seen that using the Dirac delta distribution, we converted the domain integral into a jump term. It is worth noting that herein the point ξ is treated as an internal point. Also, it has to be noted that the negative sign on the RHS of equation (33) is only for convention and could be ignored, without affecting the final formulation.

Substituting equation (34) into equation (30) gives:

$$u(\xi) + \int_\Gamma q^*(\xi, \mathrm{x}) \, u(\mathrm{x}) \, d\Gamma = \int_\Gamma U^*(\xi, \mathrm{x}) \, q(\mathrm{x}) \, d\Gamma \tag{35}$$

which is the boundary integral equation for the Laplace equation.

5 Poisson Equation

Poisson's equation is given by:

$$\nabla^2 u(\mathrm{x}) + b(\mathrm{x}) = 0 \tag{36}$$

where the non homogeneous term b(x) denotes internal sources or body forces. In order to obtain the direct integral equation for such an equation, we will follow the same steps as before. Equation (36) will be weighted using weights *U**:

$$\int_\Omega \left(\nabla^2 u(\mathrm{x}) + b(\mathrm{x})\right) U^* \, d\Omega = 0 \tag{37}$$

which can be split to give:

$$\int_\Omega \left(\nabla^2 u(\mathrm{x})\right) U^* \, d\Omega + \int_\Omega b(\mathrm{x}) \, U^* \, d\Omega = 0 \tag{38}$$

After integrating by parts twice, the first integral will lead to the same result as that of equation (35), whereas the second integral will remain unchanged. Therefore the final integral form can be obtained as follows:

$$u(\xi) + \int_\Gamma q^*(\xi, \mathrm{x}) \, u(\mathrm{x}) \, d\Gamma = \int_\Gamma U^*(\xi, \mathrm{x}) \, q(\mathrm{x}) \, d\Gamma + \int_\Omega U^*(\xi, \mathrm{x}) \, b(\mathrm{x}) \, d\Omega \tag{39}$$

The last integral on the RHS represents the *second type* of domain integrals that appear in the BEM formulation. This integral can be transformed to the boundary using many techniques (as will be discussed in future tutorials).

6 Research directions in the area of Boundary Elements

Figure 3 shows different possible areas of research in the BEM. In this primer, up to now, we have seen an introduction to exercise of three theoretical areas: how to set up the integral forms, the use of fundamental solutions, and domain integrals in BEM. Throughout this primer we will cover all of these possible areas.

7 Conclusions and future tutorials

The idea of the traditional integration by parts has been discussed and generalized. We demonstrated that this idea is based on Green's second identity. Then this idea is used to derive the direct integral forms.

In this tutorial we have covered:

1. How to set up a direct integral form for a differential equation.
2. The weighting function could be chosen to be any function, however using the two-point fundamental solutions as weighting functions reduce a certain type of domain integrals to jump terms.
3. That there are mainly two types of domain integrals in the BEM. One including the adjoint operator and the other for non-homogeneous terms.
4. Different areas of research in the BEM.

In the coming tutorial we will discuss more complex examples in setting up the direct integral equations for problems such as elasticity problems and bending of elastic plates.

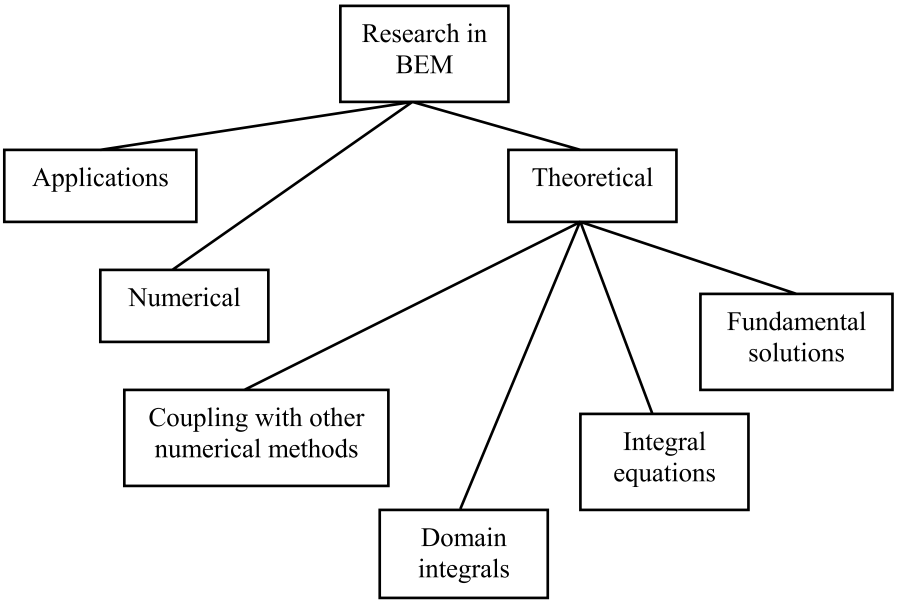

Figure 3: Research areas in the BEM.

References and Further Reading

[1] Brebbia, C.A. & Dominguez, J., Boundary Elements: An Introductory Course, WIT Press, Southampton, UK., McGraw Hill, 1992.