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Tutorial 1:
AN INTRODUCTION TO BOUNDARY ELEMENTS

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Summary and objectives

The main objective of this primer is to make the boundary element method more usable, not only among researchers but among engineers as well. This primer will provide a series of lessons and will discuss different mathematical and numerical topics of boundary element methods. In the current lesson, we will discuss:

1- An introduction and overview of the boundary element method.
2- Program testing methods and possible errors.
3- Different modeling strategies.

1 Introduction

The boundary element method is very well known among engineers and scientists. The method is proving its superiority to other numerical methods, especially if used to model an appropriate application. Despite the popularity of the boundary element method, it is not currently as popular among engineers as the finite element method; the reasons for this can be summarised as follows:

1- The complexity of the mathematical formulation.
2- The lack of existence of small computer programs.
3- The lack of teaching boundary element courses among undergraduate students.
4- The difficulty in the treatment of some numerical models, such as singularity.
5- The difficulty in modifying boundary element programs with respect to these developed using finite elements.
6- The lack of versatility of the boundary element codes.
7- The change of modelling strategy from finite elements to boundary elements.

There are two publications in the literature [1,2] by Katz and Beskos discussing why boundary elements are so difficult and presenting future developments in this field.

In this introductory lesson, we will summarise the main points of these publications and present them with a recent look at the 21st century. Also, we will show what will be discussed in future lessons.

2 Advantages and disadvantages

The Boundary Element Method (BEM), as any other numerical method, has its advantages and disadvantages. The advantages of the boundary element method are as follows:

1- Only the problem boundary needs to be discretized, which will lead to easy data preparation and less computing requirements.
2- The exact treatment of infinite and semi infinite domains.
3- The unknowns at internal locations are computed in the post processing stage, which simplifies any optimisation procedures.
4- Accurate results in the case of stress concentrations due to cracks or concentrated loadings.

On the other hand, the disadvantages of the boundary element method are as follows:

1- The system matrices are non symmetric and fully populated.
2- The fundamental solutions are not always easy to obtain.
3- The difficulty in treating slender structures.
4- The required domain discretization in case of non-linear applications.

In this primer, we will show how to make the best use of the advantages of the boundary element method; also we will show how to overcome their disadvantages. However, the most important point for the reader is that the best numerical modelling is achieved by using a coupled technique between boundary and finite elements.

3 Types of boundary element formulations

There are two main formulations of the boundary element methods: the *direct* formulation and the *indirect* formulation.

In the *direct* formulation the integral equation is formulated in terms of the unknown boundary source functions (potentials or generalized displacements) and their derivatives (fluxes or generalized tractions). The starting point for deriving the direct integral equation is either using the weighted residual statement or using the Betti-reciprocal theory. In this primer, we will show a more general and appropriate way to derive such integral equations.

Internal point functions are computed using another integral equation based on the derivatives of the first boundary integral equation. Due to the differentiations, the order of singularity in the latter integral equation is higher (usually hyper singularity). However, such singularity does not appear because such an equation is only used to compute functions at internal points. Some inaccuracies may be obtained for points close to the boundary. If the differentiated integral equation is taken to the boundary, some of its integrals should be interpreted in terms of a special notation (called Hadamard finite part integrals) and then the equation is said to be a *hyper-singular* boundary integral equation. Such new equations are very useful in modelling cracks or they could be used in accurate computation of the boundary fluxes or stresses. They could be also used to generate new equations for corner points for multiple connected domains.

The *indirect* boundary element formulation, on the other hand, is formulated by considering the superposition of the effect of fictitious flex or tractions applied on the boundary to an internal point. The problem is formulated in terms of fictitious fluxes or tractions (not in terms of the source functions and their derivatives, as that of the direct formulation) therefore it is called indirect formulation based on *fictitious fluxes or tractions* [3]. If the same formulation is carried out based on discontinuous displacements, the resulting formulation contains hyper singular integrals and is called an *indirect* formulation based on the *displacement discontinuity* method. The latter method is useful for modelling cracks as it can be considered as an equivalent case to the hyper-singular direct integral equations [4].

4 Solution of the boundary integral equation

After forming the integral equations, a numerical scheme has to be set up to solve such equations. In order to do so, the boundary of the problem is discretizaed into *elements*, where the unknown source functions (potentials or generalized displacements) are assumed to vary using polynomials (constant, linear, quadratic, etc). It can be seen that such boundary elements represent virtual discretization (not a physical one as that used in the finite element method). Inside these elements, some points called *nodes* are chosen to approximate the source functions in terms of the nodal values (via the used polynomials). In order to solve the integral equations, one can suggest that the error is set to be zero in each node. This is called the *point collocation* technique. If the overall element domain is chosen for the collocation, this is called the *continuous collocation* or *Galerkin* technique. The later gives much less error than that of the point collocation technique.

Sometimes, in the case of the direct boundary element method, the collocation nodes are placed outside the boundary to avoided singularities. In this case the method is denoted by the *regular* boundary element method and both point and continuous collocation could be used in the solution of the integral equations. This method is useful for the purpose of checking. It could be used also to model slender bodies [5] as it will be shown in a later lesson.

In the indirect boundary element method, on the other hand, if the sources are placed on an external boundary and the collocation is performed on the boundary nodes, this will avoid the singularity. After replacing the boundary integrals by summations, the method is called the *discrete collocation* method. This method is known in the literature by many different names such as: the *superposition* method, the *source-field superposition*, the *method of fundamental solution*, the *charge simulation* method, or the *modified Trefftz* method.

In this primer, we will discuss all of the above methods from their mathematical bases to the numerical implementation and we will demonstrate some solved examples.

5 Boundary element programs

The common structure of boundary element programs is described in Ref. [6]. In this primer, we will be describing a more flexible and efficient strategy for writing more general programs. Meanwhile, we will be discussing some general comments on how to test boundary element programs. Also we will discuss the different sources of errors which frequently happened in boundary element programming.

There are many types of error sources in boundary element programs, among them are the ones described in Ref. [1], which can be summarized as follows:

1- Wrong kernels: Many times the researcher picks up a kernel published in the literature. Sometimes this kernel has some kind of mistake or misprint. Therefore it is essential to check any previously published kernels. If the kernel is lengthy this checking is cumbersome. However as recommended in Ref. [1,2] symbolic computation programs could be used to perform this task. In this primer, we will describe how to derive boundary element kernels and show how to use symbolic computation packages to achieve this.

2- Integration scheme: It is very important to choose the correct integration scheme, especially for weak and near singular integrals. These two types of singularities are always ignored, in particular the later one. There are many efficient techniques used to treat such singularities. We will discuss many of these techniques such as: the non-linear coordinate transformation, the use of element subdivision, singularity isolation using Taylor expansion, etc.

3- Matrix condition: In some cases; especially when using the regular boundary element method or the superposition method (the method of the fundamental solution) the resulting system matrix is ill-conditioned. We will discuss how to deal with these cases.

4- Discretization: In some cases, especially when using the hyper-singular integral equation, to obtain good results, the problem has to be discretized with a fine mesh. Moreover in the case of using discontinuous elements or strongly graded meshes, the near singularity will have a large effect on the results. Another problem can be appeared when discretizing a curved boundary along a thin plate using straight elements. In this case the node at the element conjunction is an artificial corner and corner forces could affect the results if such corners are ignored. The same problem could appear when using curved elements with a coarse discretization.

After discussing different sources of errors that commonly appear in boundary element programs, in the coming few paragraphs, we will demonstrate an introduction to different methods of testing boundary element programs.

Unlike finite elements, it is difficult to test a boundary element program. In finite elements, only one element can be considered and checked manually. However in boundary elements, there is no physical meaning to study the case of only one element. The following examples show some outlines of some proposed testing procedures and we will consider such procedures in detail in the coming lessons. The easiest technique to check the boundary element code is to move the collocation points outside the boundary (in case of using the direct boundary element method). In this case there will be no singularity in the solution, and therefore the effect of any mistakes in the singularity subroutines can be tracked. It has to be note that in considering such a case, the near singularity has to be considered in an efficient manner.

A second check is to test a symmetric problem; the solution for this problem should yield symmetric results. In Ref. [1], Katz recommended that the results have to be checked against analytical solutions and not against any other numerical method solution.

An efficient way to check is to use analytical checking. This can be achieved by choosing a rectangular domain and discretize it into four elements, where all elements of the influence matrices can be computed manually. It has to be noted that symbolic computation packages could be used to compute the corresponding matrices for such elements. This test could be carried out using discontinuous elements to avoid column overlapping in the influence matrices, then such procedures could be repeated using continuous elements.

6 Boundary element versus finite element modelling

Apart from researchers and mathematicians, most engineers and modellers, nowadays, are more familiar with the finite element modelling. This is mainly because of the existence of extensive finite element based computer programs and packages. Due to this fact, we assume that the reader has the necessary background in finite element modelling.

The first difficulty, an engineer will face in trying to build a boundary element model is the difference in the modelling strategies between finite elements and boundary elements. Herein are some examples to show the difference in the two modelling techniques:

1- Modelling boundary conditions: commonly in finite elements any support is represented at one node or series of nodes. In boundary elements, supports are represented on the boundary using element-based tractions.
2- Modelling loads: similar to modelling boundary conditions, concentrated loads are applied at the nodes in finite elements. In boundary elements, loads are represented by continuous tractions over elements. Concentrated loads in boundary elements are possible, however they are not recommended, as they generate a singularity under the point of application; which might spoil the results. It has to be noted that the modelling of springs can be treated in a similar way.

One advantage of the boundary element method is its ability in modelling batch loading of any shape inside the domain or batch supports using their actual shapes (unlike finite elements).

Another example to show the difference in the modelling strategy is the modelling of a continuum using skeletal elements in finite elements as, for example, in the strut and tie model (the truss model) for modelling deep girders, bridge peers or pile caps or the use of grillage model for modelling of plates and roofs. Such models were developed as a way of avoiding finite element discreization. However, in the modelling of the boundary element method the thinking towards the use of the actual continuum model is more appropriate. This is mainly because the boundary element mesh is easier to generate than the skeletal models of the idealized structures.

Generally, the finite element engineer or modeller tends to idealize the structure using concentrated loading or supports. In the boundary element method, however, this strategy has to be changed. If one take deeper look, one can see that boundary element modelling is representing the actual structure from the point of view is that there is no concentrated supports or loading in reality.

7 What we need?

Katz [1] in 1987 and Beskos [2] in 1989 discussed: “what we need?” to be developed in the boundary element method. Herein, we will discuss their views and extended what needs to be developed in boundary elements in 2001:

1- A deeper insight into the mathematical and numerical back ground of the method.
2- A systematic method for the derivation of the fundamental and particular solutions.
3- Stable integration formula.
4- General-purpose programs and small programs available for engineers.
5- More efficient coupling between boundary and finite elements.
6- Undergraduate courses to teach the boundary element method to students.
7- Short courses to teach boundary element analysis to engineers.

In this primer we will discuss all of the above points and the new developments related to each.

8 Conclusions and future tutorials

It can be seen that there are many points, which have to be cleared to engineers in order for them to use the boundary element method. Moreover as recommended by Katz in Ref. [1] the boundary element researchers have to supply the information about areas which are not suitable for the method. It is the purpose of the “*Boundary Element Advisor*” section in this journal to make a network among boundary element researchers and users concerning new formulations, unsuitable areas, problems, ideas, etc.

The author believes that both sections (the *Boundary Element Primer* and the *Boundary Element Advisor*) will work together to make the boundary element method more popular among engineers and students. This will focus on increasing the level of reliability of the boundary element method among engineers and modellers for solving practical applications.

In the coming lesson we will discuss (in detail) the integral representation of the governing differential equations with several examples for potential, elasticity and plates.

References and Further Readings

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