Chapter 3

Elastostatics

3.1 Introduction

In this chapter the application of boundary elements to study linear elastostatics problems is presented. The presentation is based on the direct approach and follows the notation initiated by Alarcon, Brebbia and Dominguez in references [1] and [6], and which is consistent with the one used in previous chapters. Reference [1] gave a comprehensive treatment of higher order elements.

The direct formulation of boundary elements for elastostatics was presented by Rizzo in 1967 [2] following the work of Jaswon (see Chapter 2). The basic integral representation known as Somigliana's identity [3] was taken to the boundary which was then discretized into constant elements in a way similar to that previously used for potential problems in some of the papers presented in Chapter 2. Cruse and Rizzo [4] and Cruse [5] extended the formulation to elastodynamics.

The chapter starts with a review of the basic equations of linear elastostatics which are then used to generate the required boundary integrals. Another section deals with the fundamental solutions and explains how they can be obtained by integration of the basic governing equations. This section formulates these solutions starting with the Galerkin vector, and in this way provides a tool for obtaining fundamental solutions in many other problems.

Section 5 discusses the discretization of the body into elements and sets up the methodology to create the boundary element equations. Topics such as integration, rigid body motion and boundary conditions are contained in this section.

An important aspect of boundary elements is the correct consideration of body forces, which whenever possible should be taken to the boundary. A special section is dedicated to the treatment of these forces and how the original domain integrals can be transformed into boundary integrals, using the Galerkin vectors already defined in the part on fundamental solution.

Many engineering problems are inhomogeneous and it is then necessary to divide the body into subregions with different properties. Sometimes the subdivision is preferred for simple computational or modelling reasons. The idea is important in boundary element applications and it is discussed in detail in Section 3.7.

Although the direct formulation is usually associated with boundary elements, it may be convenient in certain cases to apply the so called indirect formulation. Section 3.8 demonstrates how these formulations can be obtained as special cases of the more general direct approach.

Section 3.9 shows how under certain conditions of geometrical and boundary conditions symmetry, a general three dimensional body can be transformed into

an axisymmetric problem, with considerable savings in computer and modelling time.

The final section in this chapter deals with the case of orthotropic and anisotropic bodies for which the fundamental solution is more difficult to formulate than for isotropic elastostatics. The section shows how these solutions can be found and sets the basis for changing isotropic boundary element codes into more general anisotropic programs.

3.2 Basic Equations of Linear Elastostatics

In what follows we will restrict our discussions to linear elasticity, i.e. problems for which one assumes that the material behaves linearly and the changes in orientation of the body in the deformed state are negligible. The latter assumption leads to linear strain displacement relations and also allows the equilibrium equations to be referred to the undeformed geometry.

We will use the indicial notation throughout in addition to the matrix notation as otherwise the formulae will become difficult to write.

In solid mechanics one needs to consider the forces or state of stress in the body and the deformations or state of strain. Both states are interrelated by applying the material behaviour or constitutive equations, which establish the relationship between stresses and strains.

State of Stress

Let us define the state of stress at a point in terms of stress components (figure 3.1). In principle one has 9 different components which can be grouped together in a stress tensor, i.e.

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{3.1}$$

These components are not all independent but are related through the equilibrium equations, which are of two types, (i) moment equations and (ii) direct components equations.

The moment equilibrium equation can be written by taking moments of the stress components with respect to a point in the differential element and in the limit produce the so called complementary shear relationships, i.e.

$$\sigma_{21} = \sigma_{12}; \qquad \sigma_{31} = \sigma_{13}; \qquad \sigma_{32} = \sigma_{23} \tag{3.2}$$

Equilibrium of the forces in the x_1, x_2 and x_3 directions produced the well known force equilibrium equations which need to be satisfied throughout the

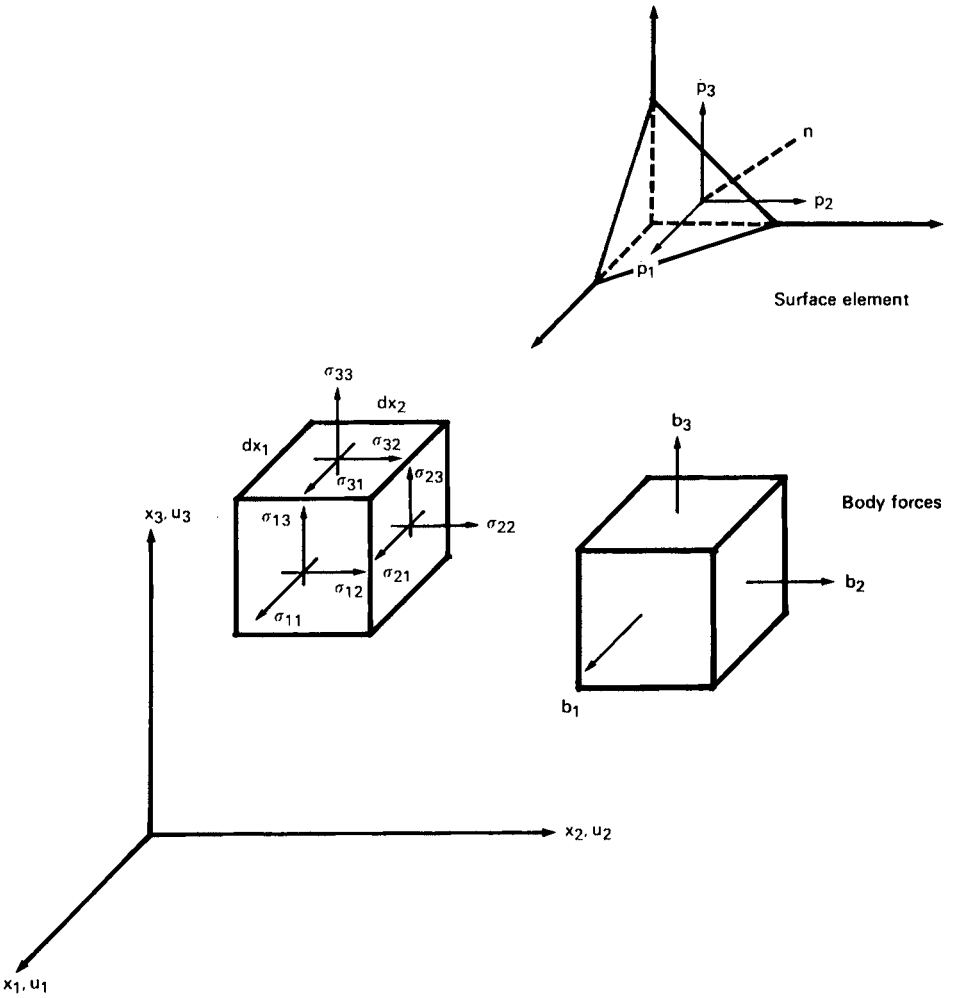

Figure 3.1 Notation for surface forces, stresses and displacements

domain (i.e. interior of the body), i.e.

$$\begin{aligned}
&\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{12}}{\partial x_2}+\frac{\partial \sigma_{13}}{\partial x_3}+b_1=0\\
&\frac{\partial \sigma_{21}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2}+\frac{\partial \sigma_{23}}{\partial x_3}+b_2=0\\
&\frac{\partial \sigma_{31}}{\partial x_1}+\frac{\partial \sigma_{32}}{\partial x_2}+\frac{\partial \sigma_{33}}{\partial x_3}+b_3=0
\end{aligned} \tag{3.3}$$

where b_i are the body forces components.

In order to write these equations in a more compact manner we will use indicial notation. The set of equations in (3.3) can be written simply as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \qquad \text{in } \Omega \tag{3.4}$$

or

$$\sigma_{ij,j} + b_i = 0$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$. Internal indices such as j vary first and then the i indices which produce three different equations. The comma indicates derivative.

The stress components are projected into a differential of the boundary $d\Gamma$ and produce surface force intensities or tractions which are denoted by p_i such that

$$\begin{aligned} p_1 &= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ p_2 &= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ p_3 &= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{aligned} \tag{3.5}$$

where $n_1 n_2 n_3$ are the direction cosines of the outward normal n with respect to the $x_1 x_2 x_3$ axis, i.e.

$$n_1 = \cos(n, x_1); \qquad n_2 = \cos(n, x_2); \qquad n_3 = \cos(n, x_3) \tag{3.6}$$

Equations (3.5) can also be written in a compact form in indicial notation,

$$p_i = \sigma_{ij} n_j \qquad \text{on } \Gamma \tag{3.7}$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$.

The tractions are assumed to be given on the Γ_2 part of the boundary and they are the 'natural' boundary conditions for this problem. Hence,

$$\left.\begin{aligned} p_1 &= \bar{p}_1 \\ p_2 &= \bar{p}_2 \\ p_3 &= \bar{p}_3 \end{aligned}\right\} \qquad \text{on } \Gamma_2 \tag{3.8}$$

These conditions imply that the applied tractions $\bar{p}_i$ have to be in equilibrium with the traction components obtained from the internal stresses at the boundary, i.e.

$$p_i = \sigma_{ij} n_j = \bar{p}_i \qquad \text{on } \Gamma_2 \tag{3.9}$$

State of Strain

The deformations of the boundary are functions of the displacements, which have the components, $u_1\ u_2\ u_3$ at every point. They produce strains which for

linear cases are

Direct Strains

$$\varepsilon_{11}=\frac{\partial u_1}{\partial x_1};\qquad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2};\qquad \varepsilon_{33}=\frac{\partial u_3}{\partial x_3} \tag{3.10}$$

Shear Strains

$$\varepsilon_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right);\qquad \varepsilon_{13}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right);\qquad \varepsilon_{23}=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)$$

These expressions can also be written in indicial notation

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{3.11}$$

where $i=1, 2, 3$ and $j=1, 2, 3$. Another way of expressing (3.11) is as follows,

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i})$$

where the comma indicates derivatives.

Sometimes the state of strain is defined using the strain components arranged in a tensor (the strain tensor), i.e.

$$\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}\end{bmatrix} \tag{3.12}$$

where $\varepsilon_{21}=\varepsilon_{12}$; $\varepsilon_{31}=\varepsilon_{13}$; $\varepsilon_{32}=\varepsilon_{23}$.

It is simpler to apply boundary conditions in terms of displacements rather than in function of strains. Hence on Γ_1 the following 'essentials' conditions can be defined

$$u_1=\bar{u}_1;\qquad u_2=\bar{u}_2;\qquad u_3=\bar{u}_3\qquad \text{on } \Gamma_1 \tag{3.13}$$

or

$$u_j=\bar{u}_j\qquad j=1, 2, 3\qquad \text{on } \Gamma_1$$

where $\bar{u}_j$ are the prescribed values. Note that the total Γ surface of the boundary is equal to $\Gamma_1+\Gamma_2$.

Constitutive Relationships

The states of stress and strains in a body are related throughout the strain-stress or constitutive equations for the material. For a linearly elastic isotropic material one can define two constants, called Lamé's constants, λ and μ which

are associated with the volumetric and shear components. Then the stress-strain relationship can be expressed as

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{3.14}$$

where δ_{ij} is the Kronecker delta ($\equiv 1$ for $i = j$ and $\equiv 0$ for $i \neq j$). Notice that ε_{kk} has only internal indexes and hence it implies a sum of the three direct strain components, and because of that is called the volumetric strain, i.e.

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{3.15}$$

The inverse of (3.14) can be written as

$$\varepsilon_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \sigma_{kk} + \frac{1}{2\mu} \sigma_{ij} \tag{3.16}$$

where $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.

The Lamé's constant can be expressed in terms of the more familiar shear modulus G, Modulus of Elasticity E and Poisson's ratio ν by the following formulae,

$$\mu = G = \frac{E}{2(1+\nu)}; \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{3.17}$$

The strain and stress in terms of E and ν can be written as

$$\varepsilon_{ij} = -\frac{\nu}{E} \sigma_{kk} \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} \tag{3.18}$$

and

$$\sigma_{ij} = \frac{E}{(1+\nu)} \left[\frac{\nu}{(1-2\nu)} \delta_{ij} \varepsilon_{kk} + \varepsilon_{ij} \right] \tag{3.19}$$

For some particular problems (specially in soil mechanics) one may prefer to use the bulk modulus K.

In these cases one defines the *deviatoric* stress and strain components

$$\sigma'_{ij} = \sigma_{ij} - \tfrac{1}{3} \sigma_{kk} \delta_{ij} \tag{3.20}$$

$$\varepsilon'_{ij} = \varepsilon_{ij} - \tfrac{1}{3} \varepsilon_{kk} \delta_{ij} \tag{3.21}$$

Thus the constitutive equations are expressed as

$$\sigma'_{ij} = 2G\varepsilon'_{ij}, \qquad p = -K\varepsilon_{kk} \tag{3.22}$$

p is the mean pressure.

$$p = -\frac{\sigma_{kk}}{3} \tag{3.23}$$

and

$$K = \lambda + \tfrac{2}{3}G = E/(3(1-2\nu)) \tag{3.24}$$

In general for isotropic elastic materials all material constants can be expressed in function of two independent constants.

The equations of equilibrium (3), strain-displacement relations (6) and constitutive equations (6) give a complete system of equations from which one can determine the components of stress (6), displacements (3) and strains (6).

Initial Stresses or Strains

In many problems one can have initial state of stress or strain due to temperature or other causes. Consider an initial strain for instance. In this case the *elastic* strain components are obtained by subtracting from the total strain those initial strains, i.e.

$$\varepsilon^{e}_{ij} = \varepsilon^{t}_{ij} - \varepsilon^{o}_{ij} \tag{3.25}$$

where ε^{e}_{ij} indicates the elastic components, ε^{t}_{ij} the total and ε^{o}_{ij} the initial strains.

One can now define the stresses using the elastic strains, i.e.

$$\sigma_{ij} = \lambda\delta_{ij}(\varepsilon^{t}_{kk} - \varepsilon^{o}_{kk}) + 2\mu(\varepsilon^{t}_{ij} - \varepsilon^{o}_{ij}) \tag{3.26}$$

or

$$\begin{aligned}\sigma_{ij} &= (\lambda\delta_{ij}\varepsilon^{t}_{kk} + 2\mu\varepsilon^{t}_{ij}) - (\lambda\delta_{ij}\varepsilon^{o}_{kk} + 2\mu\varepsilon^{o}_{ij}) \\ &= \sigma^{t}_{ij} + \sigma^{o}_{ij}\end{aligned} \tag{3.27}$$

The σ^{o}_{ij} components are called initial stresses and are defined as,

$$\sigma^{o}_{ij} = -(\lambda\delta_{ij}\varepsilon^{o}_{kk} + 2\mu\varepsilon^{o}_{ij}) \tag{3.28}$$

It will be seen later on how the initial stress or strain components can be analysed and included in the boundary integral formulation.

Notice that if the initial strains are due to temperature and the material is thermally isotropic, one can write

$$\varepsilon^{o}_{ii} = \alpha\theta \tag{3.29}$$

where α is the dilatation coefficient and θ the difference in temperature. The values of σ^{o}_{ij} are given by

$$\sigma^{o}_{ij} = -2\mu\left(\frac{1+\nu}{1-2\nu}\right)\alpha\theta\delta_{ij} \tag{3.30}$$

3.3 Fundamental Solutions

The formulation of the boundary integral equations for elastostatics to be described in section 3.4 requires the knowledge of the solution of the elastic problems with the same material properties as the body under consideration but corresponding to an infinite domain loaded with a concentrated unit point load. This is the fundamental solution of elastostatics and is due to Kelvin.

If the Equilibrium equations (3.4) are expressed in terms of displacements components one obtains Navier's equations, i.e. consider (3.4)

$$\sigma_{lj,j} + b_l = 0 \tag{3.31}$$

Substitute above the stress strain relationships (3.19), i.e.

$$\sigma_{lj} = 2\mu\left[\frac{\nu}{1-2\nu}\delta_{lj}\varepsilon_{mm} + \varepsilon_{lj}\right] \tag{3.32}$$

and the strain displacement equation (3.11)

$$\varepsilon_{lj} = \tfrac{1}{2}(u_{l,j} + u_{j,l}) \tag{3.33}$$

The results are the Navier equations or equilibrium equations in terms of displacements, that is

$$\left(\frac{1}{1-2\nu}\right)u_{j,jl} + u_{l,jj} + \frac{1}{\mu}b_l = 0 \tag{3.34}$$

Kelvin solution is obtained from equation (3.34) when a unit concentrated load is applied at a point 'i' in the direction of the unit vector e_l , i.e.

$$b_l = \Delta^i e_l \tag{3.35}$$

An easy way of computing the fundamental solution is using the representation of the displacement in terms of Galerkin's vector. One assumes a vector **G** from which the displacement components may be obtained as

$$u_j = G_{j,mm} - \frac{1}{2(1-\nu)}G_{m,jm} \tag{3.36}$$

Substitution of equations (3.35) and (3.36) into equation (3.34) gives

$$G_{l,mmjj} + \frac{1}{\mu}\Delta^i e_l = 0 \tag{3.37}$$

or

$$\nabla^2(\nabla^2 G_l) + \frac{1}{\mu}\Delta^i e_l = 0 \tag{3.38}$$

This equation may be written for three-dimensional or two-dimensional plane strain problems as

$$\nabla^2(F_l) + \frac{1}{\mu}\Delta^i e_l = 0 \tag{3.39}$$

where

$$F_l = \nabla^2 G_l \tag{3.40}$$

Notice that equation (3.39) is similar to (2.9) from which the fundamental solution for potential problems was obtained. Solution of equation (3.39) gives

$$F_l = \frac{1}{4\pi r\mu} e_l \tag{3.41}$$

for three-dimensions, and

$$F_l = \frac{1}{2\pi\mu} \ln\left(\frac{1}{r}\right) e_l \tag{3.42}$$

for two-dimensions. Substitution of equations (3.41) or (3.42) into (3.40) gives

$$\nabla^2 G_l = \frac{1}{4\pi r\mu} e_l \tag{3.43}$$

for three-dimensions and

$$\nabla^2 G_l = \frac{1}{2\pi\mu} \ln\left(\frac{1}{r}\right) e_l \tag{3.44}$$

for two-dimensions, which solutions are

$$G_l = G \cdot e_l \tag{3.45}$$

where

$$G = \frac{1}{8\pi\mu} r \tag{3.46}$$

for three-dimensions and

$$G = \frac{1}{8\pi\mu} r^2 \ln\left(\frac{1}{r}\right) \tag{3.47}$$

for two dimensions. Taking each load as independent, one can write,

$$G_{lk} = G\delta_{lk} \tag{3.48}$$

where G_{lk} is the k component of the Galerkin's vector at any point when a unit

load is applied at '*i*' in the *l* direction. The displacement at any point in the domain for the point load considering each direction as independent is written

$$u_k^* = u_{lk}^* e_l \tag{3.49}$$

where u_{lk}^* represents the displacement at any point in the k direction when a unit load is applied at '*i*' in the *l* direction.

In accordance with the definition of equation (3.36) one can now write,

$$u_{lk}^* = G_{lk,mm} - \frac{1}{2(1-\nu)} G_{lm,km} \tag{3.50}$$

Substituting equations (3.48) and (3.46) into (3.50) one obtains

$$u_{lk}^* = \frac{1}{16\pi\mu(1-\nu)r}[(3-4\nu)\delta_{lk} + r_{,l}r_{,k}] \tag{3.51}$$

for three dimensional problems. Notice that

$$r_{,l} = \frac{\partial r}{\partial x_l}; \qquad r_{,k} = \frac{\partial r}{\partial x_k} \tag{3.52}$$

These are the ratios between the projection of **r** in the $x_1x_2x_3$ directions, which we can call r_1, r_2 and r_3 and the length of **r** (see figure 3.3); i.e.

$$r_{,l} = \frac{r_l}{r} \tag{3.53}$$

For the two dimensional plane strain problem the fundamental solution in terms of displacements is obtained by substituting equations (3.48) and (3.47) into (3.50), which gives

$$u_{lk}^* = \frac{1}{8\pi\mu(1-\nu)}[(3-4\nu)\ln\frac{1}{r}\delta_{lk} + r_{,l}r_{,k}] \tag{3.54}$$

Notice that when Laplace's operator is applied to the Galerkin's vector given by equation (3.47) the computed value of F_l differs from that of equation (3.42) in a constant, the resulting value being also the solution of (3.39). Any value of G different from that given by (3.47) in r^2 terms may also be used and in fact one of those fundamental solutions of the biharmonic equation for two dimensions was used in Chapter 2 (equation (2.110)). These values produce fundamental solution displacements u^* identical to equation (3.54) except for a rigid body motion that is neglected because it does not change the solution of the problem as can be seen below.

Stresses at any internal point can be written using the strain-displacement relations (3.11) and the strain-stress equation (3.19). They can be expressed as,

$$\sigma^*_{kj} = S^*_{lkj} e_l \tag{3.55}$$

where the kernel S_{lkj} has been obtained from u^*_{lk} and will be written in full later on.

The tractions or surface forces on the Γ boundary with normal $\vec{n}$ can be written through (3.55) and equation (3.9) as,

$$p^*_k = p^*_{lk} e_l \tag{3.56}$$

where the traction components for three dimensional case are

$$p^*_{lk} = -\frac{1}{8\pi(1-\nu)r^2}\left[\frac{\partial r}{\partial n}[(1-2\nu)\delta_{lk} + 3r_{,l}r_{,k}] + (1-2\nu)(n_l r_{,k} - n_k r_{,l})\right] \tag{3.57}$$

n_l and n_k are the direction cosines of the normal with respect to x_l and x_k. $\partial r/\partial n$ is the derivative of the distance vector r with respect to the normal.

For 2 dimensional plane strain problems one has

$$p^*_{lk} = -\frac{1}{4\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}[(1-2\nu)\delta_{lk} + 2r_{,l}r_{,k}] + (1-2\nu)(n_l r_{,k} - n_k r_{,l})\right] \tag{3.58}$$

Example 3.1

Consider for instance one component of the Galerkin vector, say G_1, for the case of two dimensional elasticity. Substituting it into equation (3.36) will produce the following two displacement components (which correspond to a force acting in the direction $l = 1$), i.e.

$$\begin{aligned} u_1 &= \nabla^2 G_1 - \frac{1}{2(1-\nu)}\left\{\frac{\partial^2 G_1}{\partial x_1^2}\right\} \\ u_2 &= -\frac{1}{2(1-\nu)}\frac{\partial^2 G_1}{\partial x_2 \partial x_1} \end{aligned} \tag{a}$$

One can now substitute these components into the Navier equations (3.34) which are expanded into

$$\begin{aligned} &\nabla^2 u_1 + \left(\frac{1}{1-2\nu}\right)\left\{\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1}\right\} + \frac{\Delta^i}{\mu} e_l = 0 \\ &\nabla^2 u_2 + \left(\frac{1}{1-2\nu}\right)\left\{\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2}\right\} = 0 \end{aligned} \tag{b}$$

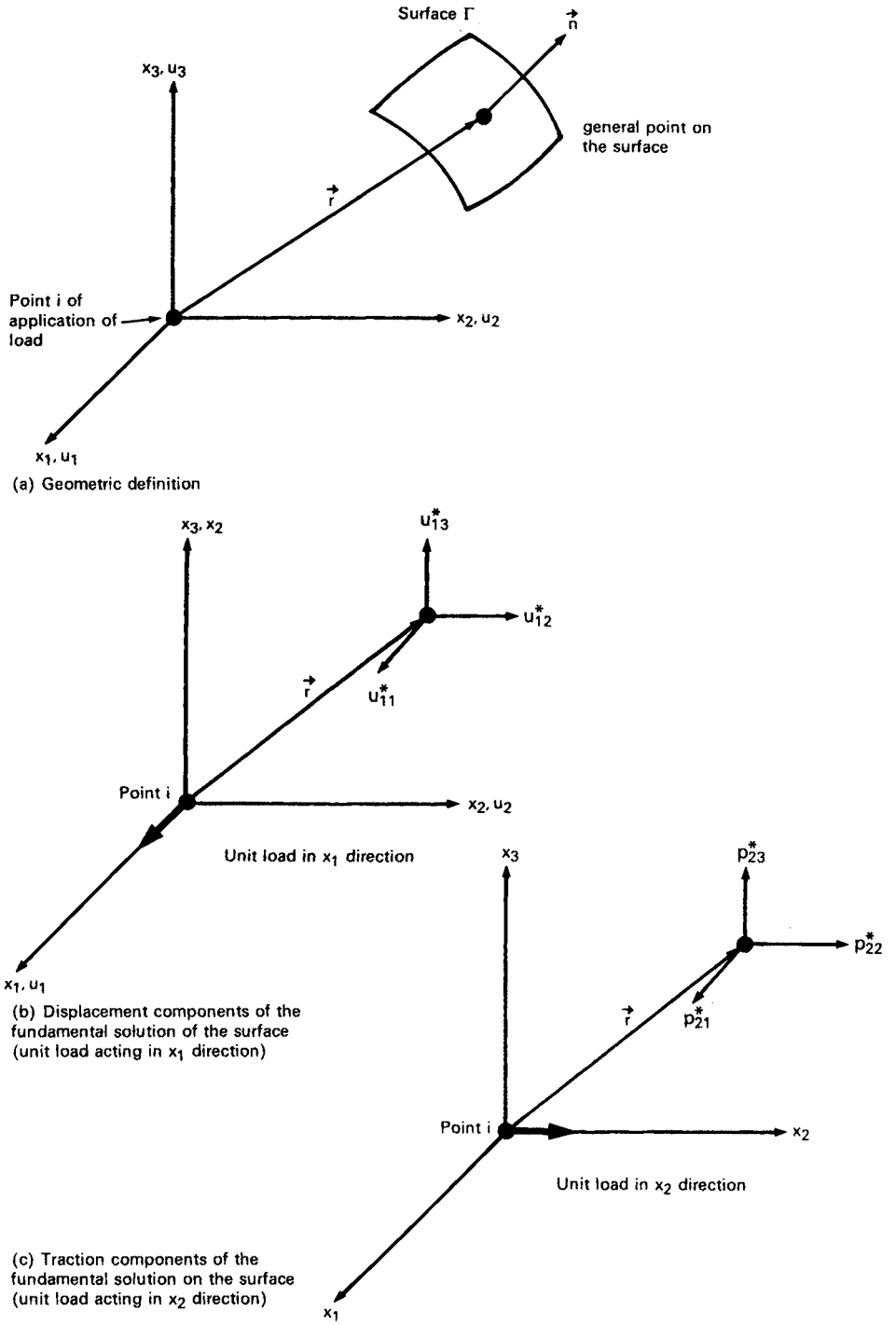

Figure 3.2 Geometrical interpretation of the components of the fundamental solution

This gives for the first equation

$$\nabla^2(\nabla^2 G_1) + \frac{\Delta^i}{\mu} e_1 = 0 \tag{c}$$

while the second is identically satisfied.

The reader can verify that the G_2 component will behave in a similar way.

3.4 Boundary Integral Formulation

The governing integral equations for elastostatics will be deduced using considerations of weighted residual. The fact that some terms which are assumed to be approximate may not be so does not detract from the use of these concepts as a general way of producing the required statement. The concepts are very similar to those used in virtual work.

Consider first that one desires to minimize the errors involved in the numerical approximation of the governing equations of elastostatics, i.e.

$$\sigma_{kj,j} + b_k = 0 \qquad \text{in } \Omega \tag{3.59}$$

which usually have to satisfy the following conditions

(i) Essential or displacement conditions

$$u_k = \bar{u}_k \qquad \text{on } \Gamma_1 \tag{3.60}$$

(ii) Natural or traction conditions

$$p_k = \sigma_{kj} n_j = \bar{p}_k \qquad \text{in } \Gamma_2 \tag{3.61}$$

Consider first that we are only interested in minimizing (3.59). To this end one can weight each of these equations by displacement type functions u_k^* and orthogonalize the product, i.e.

$$\int_\Omega (\sigma_{kj,j} + b_k) u_k^* \, d\Omega = 0 \tag{3.62}$$

If we carry out the integration by parts on the first term of this equation and group the corresponding terms together, one finds the following expression

$$-\int_\Omega \sigma_{kj} \varepsilon_{kj}^* \, d\Omega + \int_\Omega b_k u_k^* \, d\Omega = -\int_\Gamma p_k u_k^* \, d\Gamma \tag{3.63}$$

Integrating by parts one finds the adjoint of the equation (3.59), i.e.

$$\int_\Omega \sigma_{kj,j}^* u_k \, d\Omega + \int_\Omega b_k u_k^* \, d\Omega = -\int_\Gamma p_k u_k^* \, d\Gamma + \int_\Gamma p_k^* u_k \, d\Gamma \tag{3.64}$$

This expression corresponds to Betti's reciprocal theorem (notice that $\sigma^*_{kj,j} = -b^*_k$) which is sometimes used as the starting point for the boundary integral formulation.

Notice that the two terms on the right hand side are integrals on the Γ surface of the body. Let us now consider that the boundary is divided into two parts Γ_1 and Γ_2 and on each of them the boundary conditions (3.60) and (3.61) apply. Hence on can write (3.64)

$$\int_\Omega \sigma^*_{kj,j} u_k \, d\Omega + \int_\Omega b_k u^*_k \, d\Omega = -\int_{\Gamma_1} p_k u^*_k \, d\Gamma - \int_{\Gamma_2} \bar{p}_k u^*_k \, d\Gamma + \int_{\Gamma_1} \bar{u}_k p^*_k \, d\Gamma + \int_{\Gamma_2} u_k p^*_k \, d\Gamma \tag{3.65}$$

The bars represent known values of displacements u_k and tractions p_k components. One can now integrate again by parts trying to return to equation (3.62) but we will find that the resulting expression is slightly different as we have now imposed boundary conditions in Γ_1 andΓ_2. Integrating by parts twice the first integral in (3.65) one obtains,

$$\int_\Omega (\sigma_{kj,j} + b_k) u^*_k \, d\Omega = \int_{\Gamma_2} (p_k - \bar{p}_k) u^*_k \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p^*_k \, d\Gamma \tag{3.66}$$

This expression is a generalized statement that can be used to obtain the boundary integral equations. Having established this starting principle one can now return to expression (3.65) and use as weighting functions the fundamental solution presented in 3.3, which was obtained for a point load $b_l = \Delta^i$ along the direction of the unit vector e_l, i.e.

$$\sigma^*_{lj,j} + \Delta^i e_l = 0 \tag{3.67}$$

The fundamental solution may be written as before, i.e.

$$\begin{aligned} u^*_k &= u^*_{lk} e_l \\ p^*_k &= p^*_{lk} e_l \end{aligned} \tag{3.68}$$

where u^*_{lk}, p^*_{lk} are k components of displacements and tractions due to a unit point load in the l direction. The first integral in (3.65) for a particular direction e_l of the unit load becomes

$$\int_\Omega \sigma^*_{kj,j} u_k \, d\Omega = \int_\Omega \sigma^*_{lj,j} u_l \, d\Omega = -\int_\Omega \Delta^i u_l e_l \, d\Omega = -u^i_l e_l \tag{3.69}$$

where u^i_l represents the l component of the displacement at the point i of application of the load.

Equation (3.65) can now be written to represent the three separate components of the displacement at i by taking the three directions of the point load at 'i'

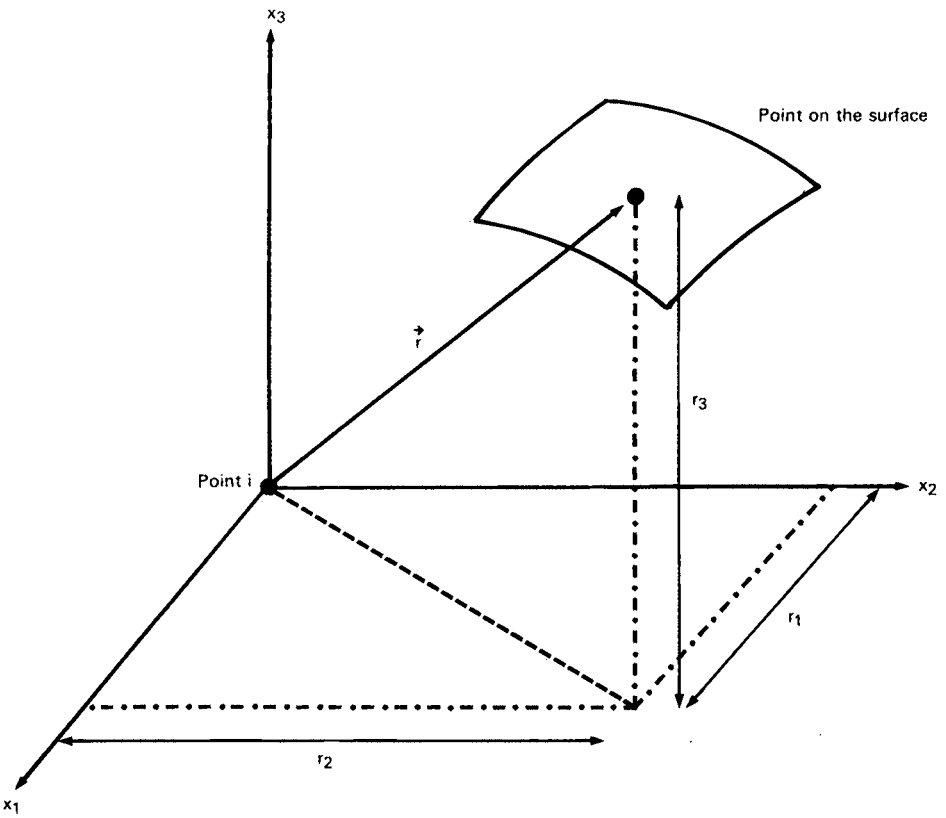

Figure 3.3 Interpretation of the components of the distance vector $\vec{r}$

independently, i.e.

$$u_l^i + \int_{\Gamma_1} p_{lk}^* \bar{u}_k \, d\Gamma + \int_{\Gamma_2} p_{lk}^* u_k \, d\Gamma$$

$$= \int_{\Gamma_1} u_{lk}^* p_k \, d\Gamma + \int_{\Gamma_2} u_{lk}^* \bar{p}_k \, d\Gamma + \int_{\Omega} u_{lk}^* b_k \, d\Omega \tag{3.70}$$

Notice that when one applies a unit point load along a particular direction 'l', the tractions and displacements at any point in the domain have components along the three or two directions (equations (3.68), and (3.51) to (3.58)) while terms of the type $\sigma_{lj,j}$ only are different from zero along the direction of the load. This leads to the fact that for a given direction 'l' at the point the first term of (3.65) only produces displacements along the direction of the load (first term of (3.70)). The rest of the terms however include products for all the components.

Equation (3.70) can be written in a more compact way if one considers the two parts of the boundary together (i.e. $\Gamma = \Gamma_1 + \Gamma_2$) and applies the boundary

conditions at a later stage. In this case (3.70) becomes

$$u_l^i + \int_\Gamma p_{lk}^* u_k \, d\Gamma = \int_\Gamma u_{lk}^* p_k \, d\Gamma + \int_\Omega u_{lk}^* b_k \, d\Omega \tag{3.71}$$

This equation is known as Somigliana's identity and gives the value of the displacements at any internal points in terms of the boundary values u_k and p_k, the forces throughout the domain and the known fundamental solution. Equation (3.71) is valid for any particular point 'i' where the forces are applied.

Boundary Points

Somigliana's identity gives the displacement at any internal point once u_k and p_k are known at every boundary point and consequently only when the boundary value problem has been solved the values at the internal points can be calculated. However, since equation (3.71) is valid for every point in Ω including Γ, a boundary integral expression can be obtained by taking (3.71) to the boundary. This expression is applied at different points on the boundary to produce a system of equations which once solved give the boundary values.

When 'i' is taken to the boundary, however, the integrals have a singularity and one needs to analyse this behaviour in the same way as it was done in Chapter 2, section 2.2 for potential problems. If we consider that the boundary is smooth at 'i' one can supplement it by a hemisphere (figure 3.4) with centre at 'i' and a small radius ε which will afterwards be taken to the limit, i.e.$\varepsilon \to 0$.

There are two types of boundary integrals in equation (3.71). Consider first the one on the right hand side and write it in function of Γ_ε surface of the hemisphere, i.e.

$$\int_\Gamma u_{lk}^* p_k \, d\Gamma = \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma - \Gamma_\varepsilon} u_{lk}^* p_k \, d\Gamma \right\} + \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} u_{lk}^* p_k \, d\Gamma \right\} \tag{3.72}$$

The first integral on the right hand side of (3.72) will simply become an integral on the whole boundary Γ when $\varepsilon \to 0$. The second integral can be written as,

$$p_k^i \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} u_{lk}^* \, d\Gamma \right\} \tag{3.73}$$

Noticing that the fundamental solution is of order $1/\varepsilon$ and the surface integral in (3.73) will produce a ε^2, one can conclude that (3.73) will tend to zero as $\varepsilon \to 0$, i.e.

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} u_{lk}^* \, d\Gamma \right\} \equiv 0 \tag{3.74}$$

In other words the integral investigated is not affected by the singularity at 'i'.

The left hand side integral in (3.71) however behaves differently. If one writes it as

$$\int_{\Gamma} p_{lk}^* u_k \, d\Gamma = \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma - \Gamma_\varepsilon} p_{lk}^* u_k \, d\Gamma \right\} + \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} p_{lk}^* u_k \, d\Gamma \right\} \tag{3.75}$$

one can see that the limit of the last integral can be written as

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} p_{lk}^* u_k \, d\Gamma \right\} = u_k^i \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} p_{lk}^* \, d\Gamma \right\} \tag{3.76}$$

The p_{lk}^* values are now of $1/\varepsilon^2$ order while the terms resulting from integration over the surface of the hemisphere are of order ε^2. Hence the integral (3.76) does not vanish when $\varepsilon \to 0$ but produces a free term. By substituting the values of p_{lk}^* as given in equation (3.57) and integrating over Γ_ε one finds

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} p_{lk}^* \, d\Gamma \right\} = -\tfrac{1}{2} \delta_{lk} \tag{3.77}$$

Hence the left hand side integral (3.75) can be written in the limit as

$$\int_{\Gamma} p_{lk}^* u_k \, d\Gamma - \tfrac{1}{2} \delta_{lk} u_k^i = \int_{\Gamma} p_{lk}^* u_k \, d\Gamma - \tfrac{1}{2} u_l^i \tag{3.78}$$

where the integral on Γ is defined in the sense of the Cauchy Principal value. This is demonstrated in detail in example 3.2 where the different terms of the fundamental solution are integrated one by one.

Therefore for boundary points equation (3.71) transforms into

$$c_{lk}^i u_k^i + \int_{\Gamma} p_{lk}^* u_k \, d\Gamma = \int_{\Gamma} u_{lk}^* p_k \, d\Gamma + \int_{\Omega} u_{lk}^* b_k \, d\Omega \tag{3.79}$$

where the integrals are in the sense of Cauchy principal value and where Γ is smooth at 'i', $c_{lk}^i = \frac{1}{2} \delta_{lk}$ When 'i' is at a point where the boundary is not smooth, the value of the integrals in equation (3.78) give different results and it is generally difficult to obtain a general expression in three dimensions.

Fortunately, explicit calculations of this value are not usually necessary as they can be obtained using rigid body motions as will be shown in section 3.5.

Boundary equation (3.79) permits to solve the general boundary value problem of elastostatics. If displacements are known over the whole boundary, equation (3.79) produces an integral equation of the first kind, if tractions are known over all the boundary an integral equation of the second kind is obtained and finally a combination of both types of boundary conditions results in a mixed integral equation.

Example 3.2

Consider the behaviour of the two types of integrals in (3.71) for the case of the point 'i' being on the smooth boundary Γ surrounded by a hemisphere as shown in figure 3.4. In the limit the radius of the hemisphere will tend to zero.

The first type of integral is as follows.

$$\int_{\Gamma} u_k p_{ik}^* \, d\Gamma = \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma - \Gamma_\varepsilon} u_k p_{ik}^* \, d\Gamma \right\} + \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} u_k p_{ik}^* \, d\Gamma \right\} \tag{a}$$

Consider the Γ_ε integral only, i.e.

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} u_k \, p_{ik}^* \, d\Gamma \right\} \\ &= \lim_{\varepsilon \to 0} \left\{ -\int_{\Gamma_\varepsilon} u_k \left[\frac{\partial r}{\partial n} \left\{ (1-2\nu)\delta_{lk} + 3 \frac{\partial r}{\partial x_l} \frac{\partial r}{\partial x_k} \right\} \right.\right. \\ &\qquad \left.\left. - (1-2\nu) \left\{ \frac{\partial r}{\partial x_l} n_k - \frac{\partial r}{\partial x_k} n_l \right\} \right] \frac{1}{8\pi(1-\nu) r^2} \right\} d\Gamma \end{aligned} \tag{b}$$

Note that $\varepsilon \equiv r$. Consider figure 3.5 where for simplicity a spherical system of coordinates is used. For this particular case the second term in equation (3.57) will disappear as,

$$\frac{\partial r}{\partial x_l} n_k - \frac{\partial r}{\partial x_k} n_l = \frac{\partial r}{\partial x_l} \frac{\partial r}{\partial x_k} - \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial x_l} \equiv 0 \tag{c}$$

Hence one only needs to consider the first term in the integral, that is,

$$I = \lim_{\varepsilon \to 0} \left\{ -\int_{\Gamma_\varepsilon} u_k \frac{\partial r}{\partial n} \left\{ (1-2\nu)\delta_{lk} + 3 \frac{\partial r}{\partial x_l} \frac{\partial r}{\partial x_k} \right\} \frac{d\Gamma}{8\pi(1-\nu) r^2} \right\} \tag{d}$$

Note that $\partial r / \partial n = 1$.

This can be expanded taking into account the geometric relationships shown in figure 3.5, for instance when $l = 1$

$$\begin{aligned} I = \lim_{\varepsilon \to 0} \left\{ -\int_{\Gamma_\varepsilon} \{ u_1^i (1-2\nu) + 3u_1^i e_1 e_1 + 3u_2^i e_1 e_2 \right. \\ \left. + 3u_3^i e_1 e_3 \} \cdot \frac{\sin\theta \, d\theta \, d\phi}{8\pi(1-\nu)} \right\} \end{aligned} \tag{e}$$

The e_i are unit vectors in the x_i direction (see figure 3.5) such that

$$e_i = n_i = \frac{\partial r}{\partial x_i} = \frac{r_i}{r}$$

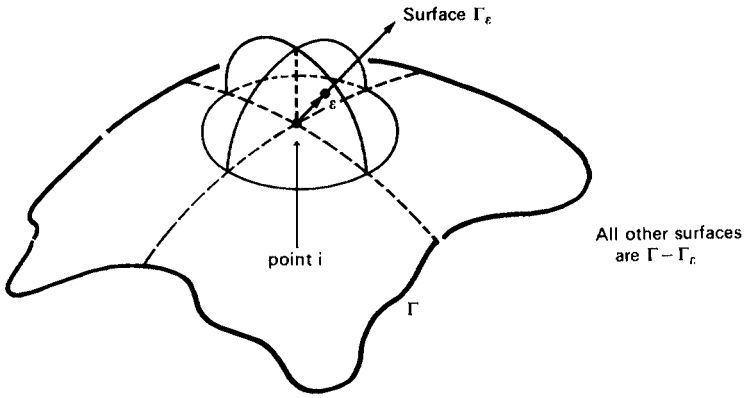

Figure 3.4 Full boundary surface Γ_ε assumed hemispherical for integration purposes

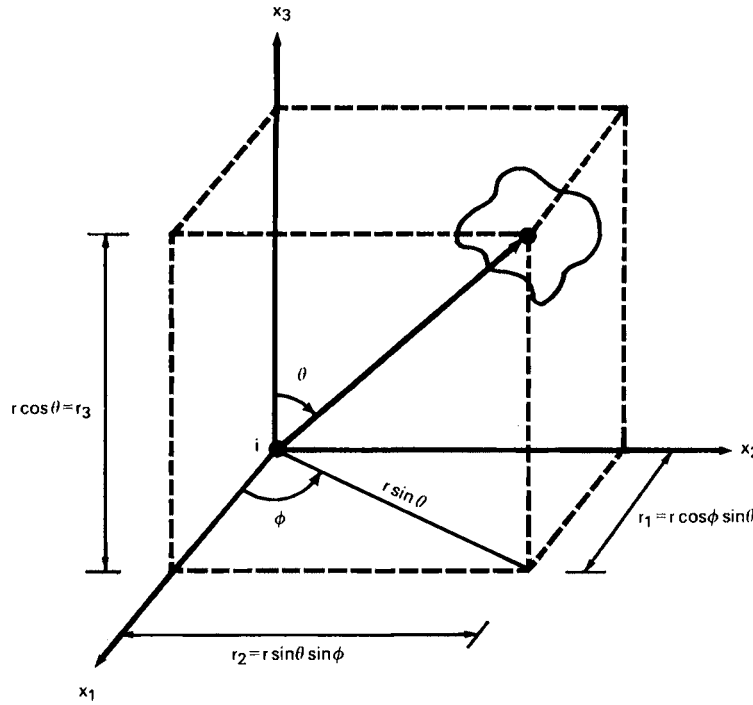

Figure 3.5 Geometry definitions

After integration we find that some of the integrals are zero and the final result is,

$$I = -\frac{1}{8\pi(1-\nu)}[(1-2\nu)2\pi + 2\pi]u_1^i = -\frac{4(1-\nu)}{8(1-\nu)}u_1^i = -\tfrac{1}{2}u_1^i \qquad \text{(f)}$$

The same can be shown to apply for $l = 2$ and $l = 3$. This result can then be written as,

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} u_k p_{lk}^* \, d\Gamma \right\} = -\frac{u_l^i}{2} \tag{g}$$

The integral

$$\int_{\Gamma} p_k u_{lk}^* \, d\Gamma \tag{h}$$

can also be written

$$\int_{\Gamma - \Gamma_\varepsilon} p_k u_{lk}^* \, d\Gamma + \int_{\Gamma_\varepsilon} p_k u_{lk}^* \, d\Gamma \tag{i}$$

but it can be easily shown that

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} p_k u_{lk}^* \, d\Gamma = 0 \tag{j}$$

and therefore this integral does not produce any new term.

Hence the final expression for the integral on a smooth boundary is,

$$\tfrac{1}{2} u_l^i + \int_{\Gamma} u_k p_{lk}^* \, d\Gamma = \int_{\Gamma} p_k u_{lk}^* \, d\Gamma + \int_{\Omega} b_k u_{lk}^* \, d\Omega \tag{k}$$

3.5 Boundary Element Formulation

In order to solve the integral equation numerically, the boundary will be discretized into a series of elements over which displacements and tractions are written in terms of their values at a series of nodal points. Writing the discretized form of (3.79) for every nodal point, a system of linear algebraic equations is obtained. Once the boundary conditions are applied the system can be solved to obtain all the unknown values and consequently an approximate solution to the boundary value problem is obtained.

It is now more convenient to work with matrices rather than carry on with the indicial notation. To this effect, one can start by defining the u and p functions which apply over each element 'j', i.e.

$$\mathbf{u} = \mathbf{\Phi} \mathbf{u}^j \tag{3.80}$$

and

$$\mathbf{p} = \mathbf{\Phi} \mathbf{p}^j \tag{3.81}$$

where $\mathbf{u}^j$ and $\mathbf{p}^j$ are the element nodal displacements and tractions, of dimensions $3 \times Q$ for three dimensions and $2 \times Q$ for two dimensions, Q being the number of

nodes on the element. **u** and **p** are the displacements and tractions at any point on the boundary Γ_e, i.e.

$$\mathbf{p} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix}; \qquad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{3.82}$$

The interpolation function matrix $\mathbf{\Phi}$ is a $3 \times 3Q$ (or $2 \times 2Q$ two dimensions) array of shape functions, i.e.

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1 & 0 & 0 & \phi_2 & 0 & 0 & \dots & \phi_Q & 0 & 0 \\ 0 & \phi_1 & 0 & 0 & \phi_2 & 0 & \dots & 0 & \phi_Q & 0 \\ 0 & 0 & \phi_1 & 0 & 0 & \phi_2 & \dots & 0 & 0 & \phi_Q \end{bmatrix}$$

$$= [\phi_1 \quad \phi_2 \quad \dots \quad \phi_Q] \tag{3.83}$$

These functions are similar to the standard type function discussed in section 2.7 (see figure 3.6). They can also be discontinuous.

Notice that the body forces at any point on the Ω domain can be expressed in vector form in function of the three components, i.e.

$$\mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{3.84}$$

The fundamental solution coefficients can be expressed as,

$$\mathbf{p}^* = \begin{bmatrix} p^*_{11} & p^*_{12} & p^*_{13} \\ p^*_{21} & p^*_{22} & p^*_{23} \\ p^*_{31} & p^*_{32} & p^*_{33} \end{bmatrix} = \begin{array}{l} \text{matrix whose coefficients, } p^*_{lk}, \\ \text{are the tractions in } k \text{ direction} \\ \text{due to a unit force at '}i\text{' acting} \\ \text{in the '}l\text{' direction} \end{array} \tag{3.85}$$

$$\mathbf{u}^* = \begin{bmatrix} u^*_{11} & u^*_{12} & u^*_{13} \\ u^*_{21} & u^*_{22} & u^*_{23} \\ u^*_{31} & u^*_{32} & u^*_{33} \end{bmatrix} = \begin{array}{l} \text{matrix whose coefficients } u^*_{lk} \text{ are} \\ \text{the displacements in the '}k\text{' direction} \\ \text{due to a unit force at '}i\text{' acting on} \\ \text{the '}l\text{' directon} \end{array} \tag{3.86}$$

With this notation equation (3.79) valid for each i point can be rewritten as follows,

$$c^i \mathbf{u}^i + \int_\Gamma \mathbf{p}^* \mathbf{u} \, d\Gamma = \int_\Gamma \mathbf{u}^* \mathbf{p} \, d\Gamma + \int_\Omega \mathbf{u}^* \mathbf{b} \, d\Omega \tag{3.87}$$

where $c^i = \frac{1}{2}$ for smooth boundary. Otherwise it will be a 3×3 (or 2×2 in two dimensions) array.

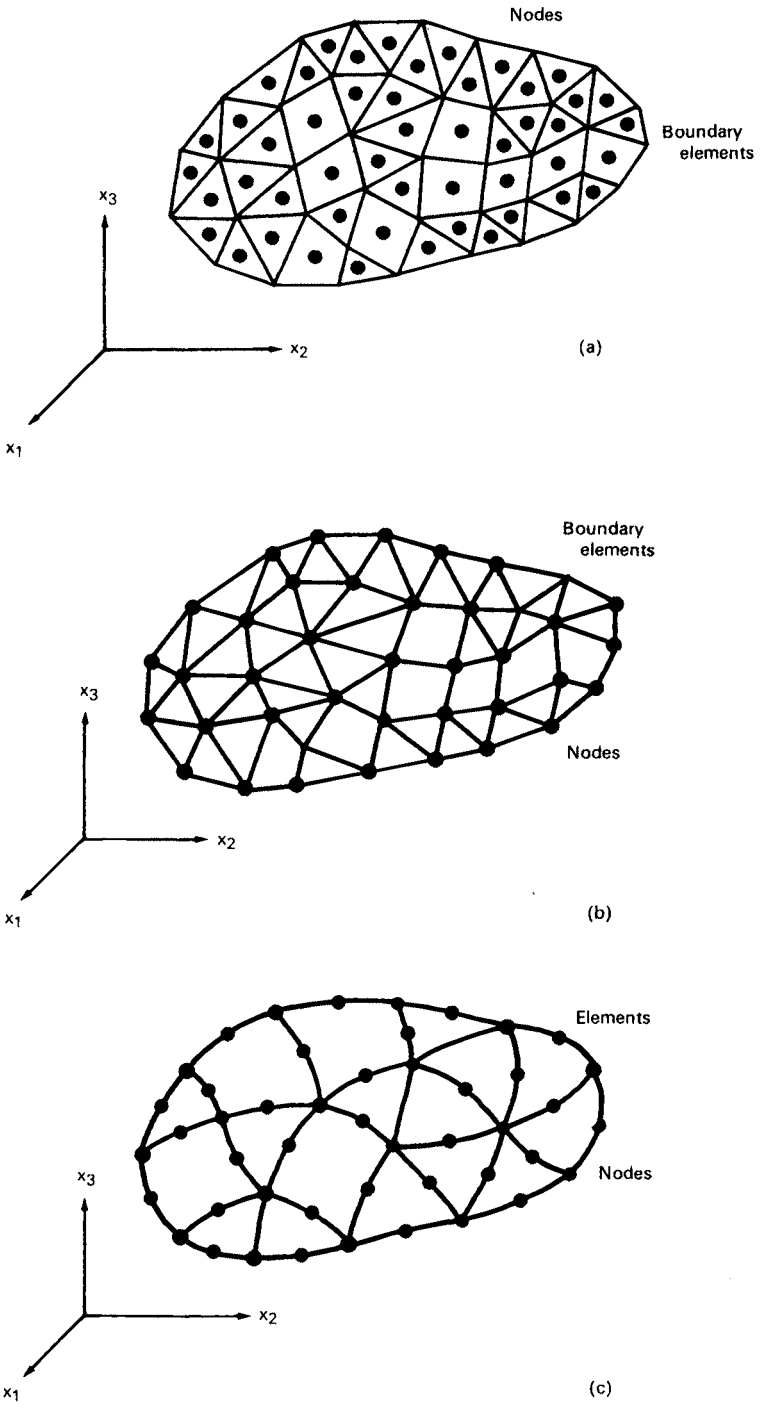

Figure 3.6 Three-dimensional body divided into (a) constant boundary elements, (b) linear boundary elements and (c) quadratic boundary elements

Notice that the Cartesian coordinates of the boundary may also be written in terms of nodal coordinates to define curved elements. If this is the case we will need to transform from one to another system and this transformation will involve introducing a Jacobian as it was shown in section 2.7. We will discuss this in what follows.

Consider now that we substitute the above interpolation functions into equation (3.87) and discretize the boundary obtaining the following equation for a nodal point.

$$\mathbf{c}^i\mathbf{u}^i + \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{p}^*\boldsymbol{\Phi}\, d\Gamma \right\} \mathbf{u}^j = \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{u}^*\boldsymbol{\Phi}\, d\Gamma \right\} \mathbf{p}^j + \sum_{s=1}^{M} \left\{ \int_{\Omega_s} \mathbf{u}^*\mathbf{b}\, d\Omega \right\} \tag{3.88}$$

Note that summation for j =1 to NE indicates summation over all the NE elements on the surface and Γ_j is the surface of a 'j' element. $\mathbf{u}^j$ and $\mathbf{p}_j$ are the nodal displacement and tractions in the element 'j'.

Notice that we have considered that the domain is divided into M internal cells over which the body forces integrals are to be computed. These domain regions over which generally a numerical integration is carried out can in some cases be avoided by taking the body force integrals to the boundary as will be seen later.

The integrals in (3.88) are usually solved numerically, particularly if the elements are curved, as in these cases it is then difficult to integrate them analytically. The interpolation functions $\boldsymbol{\Phi}$ tend to be expressed in a homogeneous system of coordinates such as those described in section 2.7 and of the type drawn in figure 3.7. The coordinates need then to be transferred from the ζ_i system to the global x_i system.

Transformation of Coordinates

The transformation of coordinates is identical to the one described in section 2.10 where two types of Jacobian were found.

(i) *Volume to Volume Transformation* relating derivatives in x_i system to those in the $\xi_1\xi_2\eta$ (figure 3.7)

$$d\Omega = |\mathbf{J}|\, d\xi_1\, d\xi_2\, d\eta \tag{3.89}$$

(ii) *Surface to Volume Transformation* which produces

$$d\Gamma = |\mathbf{G}|\, d\xi_1\, d\xi_2 \tag{3.90}$$

To compute the values of these Jacobians one needs to know the variation of x_i coordinates in function of the homogeneous system $\xi_1\xi_2\eta$, which are given in function of the same interpolation functions as used for displacements and tractions (equations (3.80) and (3.81)), i.e.

$$\mathbf{x} = \boldsymbol{\Phi}\mathbf{x}^j \tag{3.91}$$

$\mathbf{x}^j$ are the nodal values of the coordinates over the element under consideration,

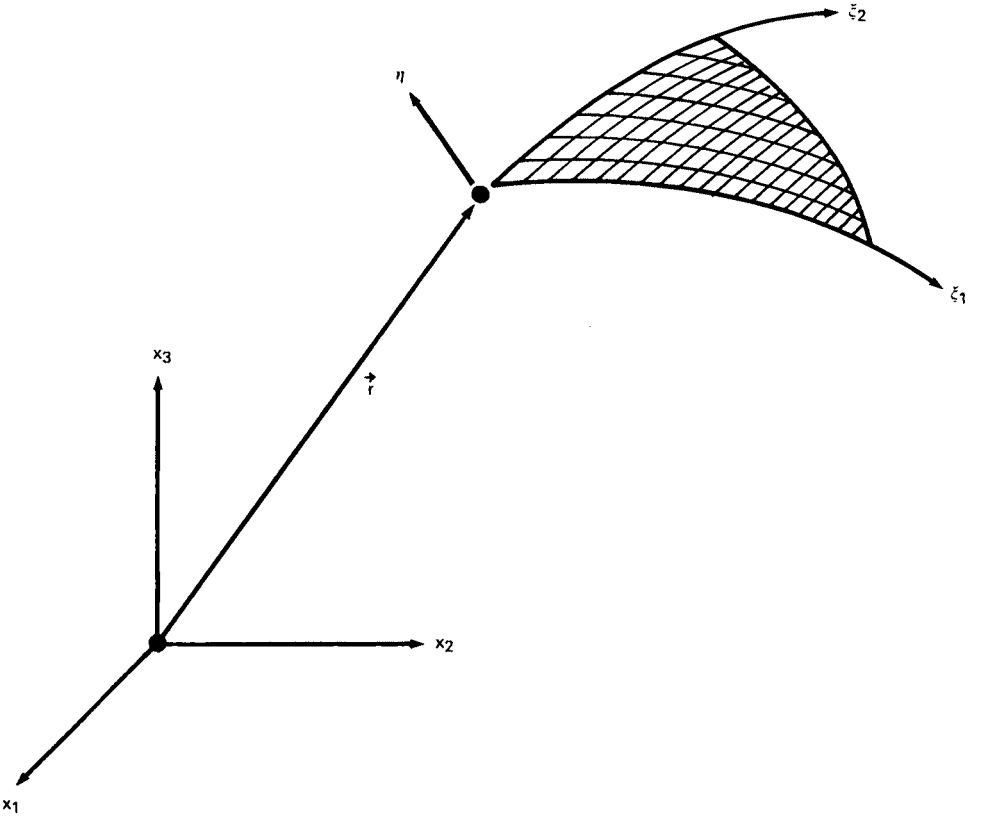

Figure 3.7 Coordinate systems for an element on a curved surface

and ϕ is the same interpolation function used for displacements and tractions as given in equation (3.83).

Equation (3.88) can now be written as

$$\mathbf{c}^i\mathbf{u}^i + \sum_{j=1}^{NE}\left\{\int_{\Gamma_j}\mathbf{p}^*\boldsymbol{\Phi}|\mathbf{G}|\,d\xi_1\,d\xi_2\right\}\mathbf{u}^j = \sum_{j=1}^{NE}\left\{\int_{\Gamma_j}\mathbf{u}^*\boldsymbol{\Phi}|\mathbf{G}|\,d\xi_1\,d\xi_2\right\}\mathbf{p}^j + \sum_{s=1}^{M}\left\{\int_{\Omega_s}\mathbf{u}^*\mathbf{b}|\mathbf{J}|\,d\xi_1\,d\xi_2\,d\eta\right\} \tag{3.92}$$

Applying numerical integration to the above formula (see Appendix A) one obtains,

$$\mathbf{c}^i\mathbf{u}^i + \sum_{j=1}^{NE}\left\{\sum_{k=1}^{l} w_k(\mathbf{p}^*\boldsymbol{\Phi})_k|\mathbf{G}|\right\}\mathbf{u}^j = \sum_{j=1}^{NE}\left\{\sum_{k=1}^{l} w_k(\mathbf{u}^*\boldsymbol{\Phi})_k|\mathbf{G}|\right\}\mathbf{p}^j + \sum_{s=1}^{M}\left\{\sum_{p=1}^{r} w_p(\mathbf{u}^*\mathbf{b}^*)_p|\mathbf{J}|\right\} \tag{3.93}$$

where l is the number of integration points on the surface elements and w_k the weight at those points. r is the number of integration points on the cells. Functions such as $(\mathbf{p}^*\boldsymbol{\Phi})$, $(\mathbf{u}^*\boldsymbol{\Phi})$ and $(\mathbf{u}^*\mathbf{b})$ have to be evaluated at the integration points.

System of Equations

Equation (3.88) or its numerical integral form (3.93) correspond to a particular node 'i' and once integrated can be written as,

$$\mathbf{c}^i\mathbf{u}^i + \sum_{j=1}^{N} \hat{\mathbf{H}}^{ij}\mathbf{u}^j = \sum_{j=1}^{N} \mathbf{G}^{ij}\mathbf{p}^j + \sum_{s=1}^{M} \mathbf{B}^{is} \tag{3.94}$$

where N is the number of nodes, $\mathbf{u}^j$ and $\mathbf{p}^j$ are the displacements and tractions at node 'j'. The influence matrices $\mathbf{H}$ and $\mathbf{G}$ (now 3×3 for three dimensions and 2×2 in two dimensional cases) are

$$\begin{aligned} \hat{\mathbf{H}}^{ij} &= \sum_t \int_{\Gamma_t} \mathbf{p}^*\boldsymbol{\Phi}_q \, d\Gamma \\ \mathbf{G}^{ij} &= \sum_t \int_{\Gamma_t} \mathbf{u}^*\boldsymbol{\Phi}_q \, d\Gamma \end{aligned} \tag{3.95}$$

where the summation extends to all the elements to which node 'j' belongs and q is number of order of the node 'j' within element t. For constant elements, the summation extends only to one element, $t \equiv j$ and Φ_q is the identity matrix.

$$\mathbf{B}^{is} = \int_{\Omega_s} \mathbf{u}^*\mathbf{b} \, d\Omega$$

Calling

$$\begin{aligned} \mathbf{H}^{ij} &= \hat{\mathbf{H}}^{ij} \qquad \text{if } i \neq j \\ \mathbf{H}^{ij} &= \hat{\mathbf{H}}^{ij} + \mathbf{c}^i \qquad \text{if } i = j \end{aligned} \tag{3.96}$$

equation (3.94) for node 'i' becomes,

$$\sum_{j=1}^{N} \mathbf{H}^{ij}\mathbf{u}^j = \sum_{j=1}^{N} \mathbf{G}^{ij}\mathbf{p}^j + \sum_{s=1}^{M} \mathbf{B}^{is} \tag{3.97}$$

The contribution for all 'i' nodes can be written together in matrix form to give the global system equations, i.e.

$$\mathbf{HU} = \mathbf{GP} + \mathbf{B} \tag{3.98}$$

Notice that the elements $\mathbf{c}^i$ will be a series of 3×3 submatrices on the diagonal $\mathbf{H}$ (or 2×2 in two dimensional cases). The elements of these submatrices are not simply given by the solid angle but can become very cumbersome to compute

analytically. Fortunately this is not required as they can be found by consideration of rigid body movement as we will see shortly.

The vectors **U** and **P** represent all the values of displacements and tractions before applying boundary conditions. These conditions can be introduced by rearranging the columns in **H** and **G**, passing all unknowns to a vector **X** on the left hand side. This gives the final system of equations, i.e.

$$\mathbf{AX} = \mathbf{F} \tag{3.99}$$

Notice that the **B** vector has been incorporated in **F**. Solving the above system all boundary values are fully determined.

Rigid Body Considerations

As it was pointed out the diagonal submatrices $\mathbf{H}^{ii}$ in **H** include terms in $\hat{\mathbf{H}}^{ii}$ and $\mathbf{c}^i$. Difficulties appear when trying to compute explicitly these terms particularly at corners due to the singularity of the fundamental solution. Assuming a rigid body displacement in the direction of one of the cartesian coordinates the traction and body force vector must be zero and hence from (3.98)

$$\mathbf{HI}^q = \mathbf{0} \tag{3.100}$$

where $\mathbf{I}^q$ is a vector that for all nodes has unit displacement along the 'q' direction ($q = 1$, 2 or 3) and zero displacement in any other direction. Since (3.100) has to be satisfied for any rigid body displacement one can write,

$$\mathbf{H}^{ii} = -\sum_{j=1}^{N} \mathbf{H}^{ij} \qquad (\text{for } j \neq i) \tag{3.101}$$

which gives the diagonal submatrices in terms of the rest of the terms of the **H** matrix.

The above considerations are strictly valid for closed domains. When dealing with infinite or semi-infinite regions equation (3.101) must be modified. If the rigid body displacement is prescribed for a boundless domain the integral

$$\int_{\Gamma_\infty} \mathbf{p}^* \mathbf{I}^q \, d\Gamma = \left\{ \int_{\Gamma_\infty} \mathbf{p}^* \, d\Gamma \right\} \mathbf{I}^q \tag{3.102}$$

over the external boundary Γ_∞ at infinity will not be zero and since the tractions $\mathbf{p}^*$ are due to a point load, this integral must be,

$$\int_{\Gamma_\infty} \mathbf{p}^* \, d\Gamma = -\mathbf{I} \tag{3.103}$$

where $\mathbf{I}$ is the 3×3 (or 2×2 in two dimensional cases) identity matrix. The diagonal submatrices for this case are,

$$\mathbf{H}^{ii} = \mathbf{I} - \sum_{j=1}^{N} \mathbf{H}^{ij} \qquad (\text{for } j \neq i) \tag{3.104}$$

Internal Points

Somigliana's identity (3.71) gives the displacement at any internal point in terms of the boundary displacements and tractions. Considering again its integral representation as in (3.88) one has,

$$\mathbf{u}^i = \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{u}^* \boldsymbol{\Phi} \, d\Gamma \right\} \mathbf{p}^j - \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{p}^* \boldsymbol{\Phi} \, d\Gamma \right\} \mathbf{u}^j + \sum_{s=1}^{M} \left\{ \int_{\Omega_s} \mathbf{u}^* \mathbf{b} \, d\Omega \right\} \tag{3.105}$$

where Γ_j is once more the surface corresponding to element j and 'i' is now an internal point. The internal point displacements in terms of the nodal displacements and tractions can be written in the same way as (3.94), i.e.

$$\mathbf{u}^i = \sum_{j=1}^{N} \mathbf{G}^{ij} \mathbf{p}^j - \sum_{j=1}^{N} \hat{\mathbf{H}}^{ij} \mathbf{u}^j + \sum_{s=1}^{M} \mathbf{B}^{is} \tag{3.106}$$

The terms $\mathbf{G}^{ij}$ and $\mathbf{H}^{ij}$ consist of integrals over the elements to which node j belongs. Those integrals do not contain any singularity and can be easily computed using numerical integration. Terms like $\mathbf{B}^{is}$ however will contain a singularity (notice that they are domain terms and the point i is now in the domain) and special care should be taken when computing them numerically. Being domain integrals however their order of singularity is one less than the integrals on the boundary and consequently can be more accurately computed using numerical integration formulae.

For an isotropic medium the internal stresses can be computed by differentiating the displacements at internal points and introducing the corresponding strains into the stress-strain relationships, i.e.

$$\sigma_{ij} = \frac{2\mu\nu}{1-2\nu} \delta_{ij} \frac{\partial u_l}{\partial x_l} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{3.107}$$

After carrying out the derivatives inside the integral equations one obtains,

$$\sigma_{ij} = \int_\Gamma \left\{ \frac{2\mu\nu}{1-2\nu}\delta_{ij}\frac{\partial u^*_{lk}}{\partial x_l} + \mu\left(\frac{\partial u^*_{ik}}{\partial x_j} + \frac{\partial u^*_{jk}}{\partial x_i}\right)\right\} p_k \, d\Gamma$$
$$+ \int_\Omega \left\{ \frac{2\mu\nu}{1-2\nu}\delta_{ij}\frac{\partial u^*_{lk}}{\partial x_l} + \mu\left(\frac{\partial u^*_{ik}}{\partial x_j} + \frac{\partial u^*_{jk}}{\partial x_i}\right)\right\} b_k \, d\Omega$$
$$- \int_\Gamma \left\{ \frac{2\mu\nu}{1-2\nu}\delta_{ij}\frac{\partial p^*_{lk}}{\partial x_l} + \mu\left(\frac{\partial p^*_{ik}}{\partial x_j} + \frac{\partial p^*_{jk}}{\partial x_i}\right)\right\} u_k \, d\Gamma \tag{3.108}$$

All derivatives are taken at the internal point under consideration, which is the point of application of the fundamental solution. Taking the corresponding derivatives of the fundamental solution the above equation can be written, in a compact form, as,

$$\sigma_{ij} = \int_\Gamma D_{kij} p_k \, d\Gamma - \int_\Gamma S_{kij} u_k \, d\Gamma + \int_\Omega D_{kij} b_k \, d\Omega \tag{3.109}$$

where the third order tensor components D_{kij} and S_{kij} are

$$D_{kij} = \frac{1}{r^\alpha}\{(1-2\nu)\{\delta_{ki}r_{,j} + \delta_{kj}r_{,i} - \delta_{ij}r_{,k}\}$$
$$+ \beta r_{,i}r_{,j}r_{,k}\}\frac{1}{4\alpha\pi(1-\nu)} \tag{3.110}$$

$$S_{kij} = \frac{2\mu}{r^\beta}\left\{\beta\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ik}r_{,j} + \delta_{jk}r_{,i}) - \gamma r_{,i}r_{,j}r_{,k}]\right.$$
$$+ \beta\nu(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k})$$
$$\left. + (1-2\nu)(\beta n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}) - (1-4\nu)n_k\delta_{ij}\right\}\frac{1}{4\alpha\pi(1-\nu)} \tag{3.111}$$

The above formulae are applicable for 2 or 3 dimensions. For the former case $\alpha = 1$, $\beta = 2$ and $\gamma = 4$ and for the latter $\alpha = 2$, $\beta = 3$ and $\gamma = 5$.

All the derivatives indicated by commas are taken at the boundary point x_i^B of figure 3.8, i.e.

$$r_{,i} = \frac{\partial r}{\partial x_i} = \frac{r_i^B}{r^B} \tag{3.112}$$

This derivative is equal and opposite in sign to those taken at an internal point.

Equation (3.109) is discretized by dividing the Γ boundary into a summation over all the element surfaces and assuming the corresponding interpolation functions for u_k and p_k.

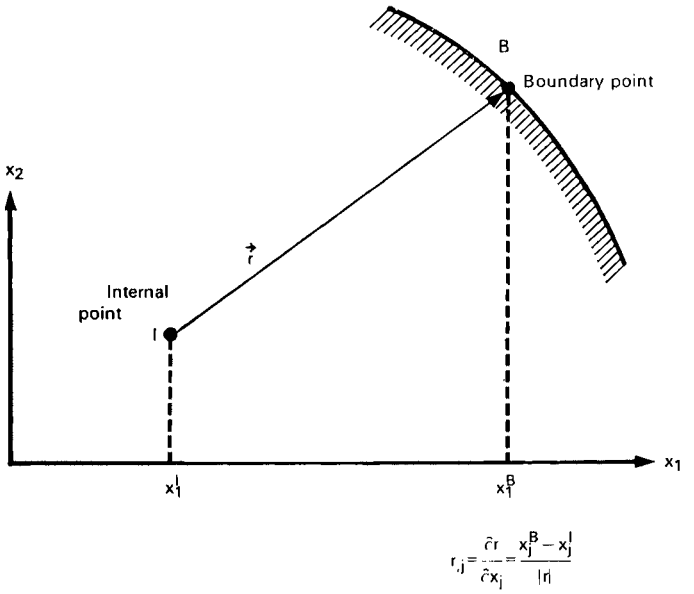

Figure 3.8 Definition of derivatives required in the internal stress formulae (3.109)

The values obtained for the internal stresses using the above formulae are in general more accurate than those computed using other numerical methods and similar discretization. The same can be said of the internal displacements computed through (3.105). However when the internal point is very close to the boundary (say less than 1/4 of the smallest length of the nearest element) because of the peak in the fundamental solution, special numerical integration schemes have to be used to obtain accurate stresses and displacements.

Notice also that the values of displacements at the boundary are known from the integral equation solution. The same can not be said of the stresses since the stress vector has more independent terms than the traction vector. The problem will be studied in the next section.

Stresses on the Boundary

The numerical solution of the governing equations produces all the boundary displacements and tractions. In many practical applications however the boundary stresses rather than the tractions are required. One possibility would be to take (3.109) to the boundary but this produces higher order singularity of a higher order and which are more difficult to compute. For three dimensional problems, the D_{kij} and S_{kij} terms contain singularities of order $1/r^2$ and $1/r^3$ respectively[*] and taking equation (3.109) to the boundary requires special consideration of how the principal values should be computed.

The simplest way of determining the stress tensor at boundary points is to compute its components from the known boundary tractions and displacements. Let us assume a three dimensional case and a local cartesian system of coordinates at the boundary points where the stresses are to be computed (see figure 3.9). It is easy to see that,

$$\begin{aligned} \sigma'_{13} &= \sigma'_{31} = p'_1 \\ \sigma'_{23} &= \sigma'_{32} = p'_2 \\ \sigma'_{33} &= p'_3 \end{aligned} \tag{3.113}$$

where the dash indicates local coordinates. In addition to the tractions a discrete expansion for the boundary displacements over the element is also known as given by (3.80), i.e.

$$\mathbf{u}' = \mathbf{R}^T \mathbf{\Phi} \mathbf{u}^j \tag{3.114}$$

where $\mathbf{R}$ is a transformation matrix from global to local system. Four components of the strain tensor can be computed by differentiating u' as follows

$$\varepsilon'_{ij} = \frac{1}{2}\left(\frac{\partial u'_j}{\partial x'_i} + \frac{\partial u'_i}{\partial x'_j}\right) \qquad i, j = 1, 2 \tag{3.115}$$

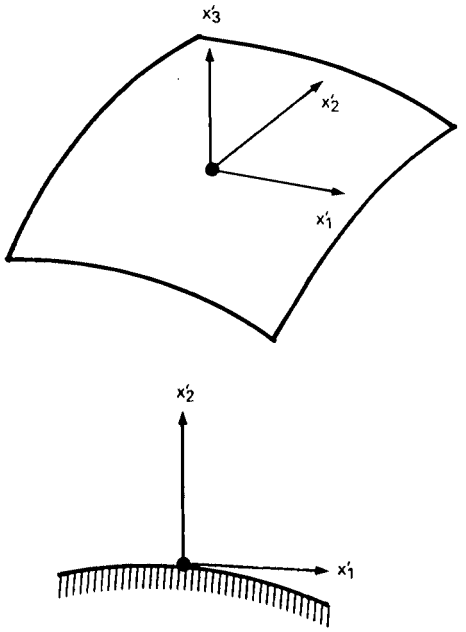

Figure 3.9 Local system of coordinates over the element

Notice that the ε'_{ij} will depend on the derivatives of the shape functions and the nodal displacements. If constant elements are used, the displacement derivatives can be computed using a finite difference approximation between adjacent nodes.

The rest of the terms of the stress tensor can now be computed from the constitutive equations as follows,

$$\begin{aligned}
\sigma'_{12} &= \sigma'_{21} = 2\mu\varepsilon'_{12} \\
\sigma'_{11} &= \frac{1}{1-\nu}\left[\nu\sigma'_{33} + 2\mu(\varepsilon'_{11} + \nu\varepsilon'_{22})\right] \\
\sigma'_{22} &= \frac{1}{1-\nu}\left[\nu\sigma'_{33} + 2\mu(\varepsilon'_{22} + \nu\varepsilon'_{11})\right]
\end{aligned} \tag{3.116}$$

For two dimensional problems the procedure is analogous. Three components of the stress tensor are obtained from the tractions, i.e.

$$\begin{aligned}
\sigma'_{12} &= \sigma'_{21} = p'_1 \\
\sigma'_{22} &= p'_2
\end{aligned} \tag{3.117}$$

and one component of the stress tensor is computed from the surface displacements as follows,

$$\varepsilon'_{11} = \frac{\partial u'_1}{\partial x'_1} \tag{3.118}$$

and the last stress component is computed from

$$\sigma'_{11} = \frac{1}{1-\nu}(\nu\sigma'_{22} + 2\mu\varepsilon'_{11}) \tag{3.119}$$

using the stress-strain relationships corresponding to plane strain.

Traction Discontinuities at Corner Points

When a node is located at a point where the boundary is not smooth, i.e. has a corner for two dimensional problems or corners and edges in three dimensional cases, a discontinuity in the traction will occur at that node. This implies that if the nodal tractions are unknown the number of equations at that node is smaller than the number of unknowns.

In order to explain what occurs, let us consider a two dimensional corner for simplicity (figure 3.10). When the tractions are known at both sides of the corner node, only the two components of the nodal displacements are unknown and no special treatment of the corner node is required. It may also happen, for any of the two components, that the displacement and one of the tractions, either 'before'

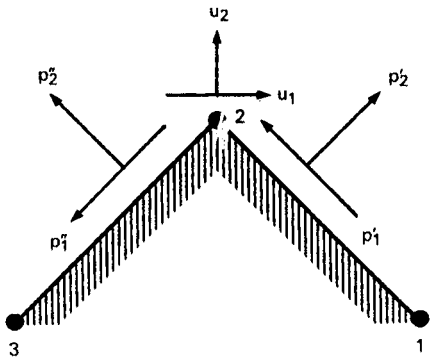

Figure 3.10 Continuous corner node

or 'after' the node, is known; then the other traction, 'after' or 'before' the node will be the unknown and the problem is solved without difficulties as will be seen in sections 4.7 and 4.8. However, when two different values ('before' and 'after' the node) of any component of the traction are unknown and only the displacement is known a special treatment of the corner is required.

The easy way then of solving the problem is by duplicating the corner node. The geometry of the problem is slightly modified and only two traction components are assigned to each node (figure 3.11(a)). The problem may now be solved by the standard procedure. The distance between the two corner nodes must be very small and it is limited by the numerical problems that may be originated by the existence of two sets of equations whose coefficients are very close to each other. In practice excellent results are obtained if the distances are not too small. When the corner node is duplicated a small gap may be left between the two nodes (figure 3.11(a)) or a small element may be assumed between the two (figure 3.11(b)). In the latter case, tractions over the small elements are assumed to be p'_1, p'_2 and p''_1, p''_2 for nodes 2′ and 2″ respectively.

Another version of the double node approach is the use of discontinuous elements already presented for potential problems in section 2.7. This consists of displacing inside the element the nodes that meet or that would meet at corners or edges (figure 3.11(c)). The approach is very simple and effective and has the added advantage that it can model better corners with high stress concentrations. When used to model singularities – such as in fracture mechanics problems – the results using discontinuous elements have converged well to the correct solution. The discontinuous elements formulation for two-dimensional linear elements will be presented in section 4.6.

The problem of having more than two unknowns at a corner can also be solved as follows. If the tractions at both sides of a corner are unknown the displacements along the two elements to which the node belongs will be known. In such a case the displacement derivatives along those elements can be obtained by derivation of the shape functions. Since the two elements follow different directions all the components of the strain tensor at the corner can be written in terms of nodal displacements of those elements and the stress tensor and any traction at the

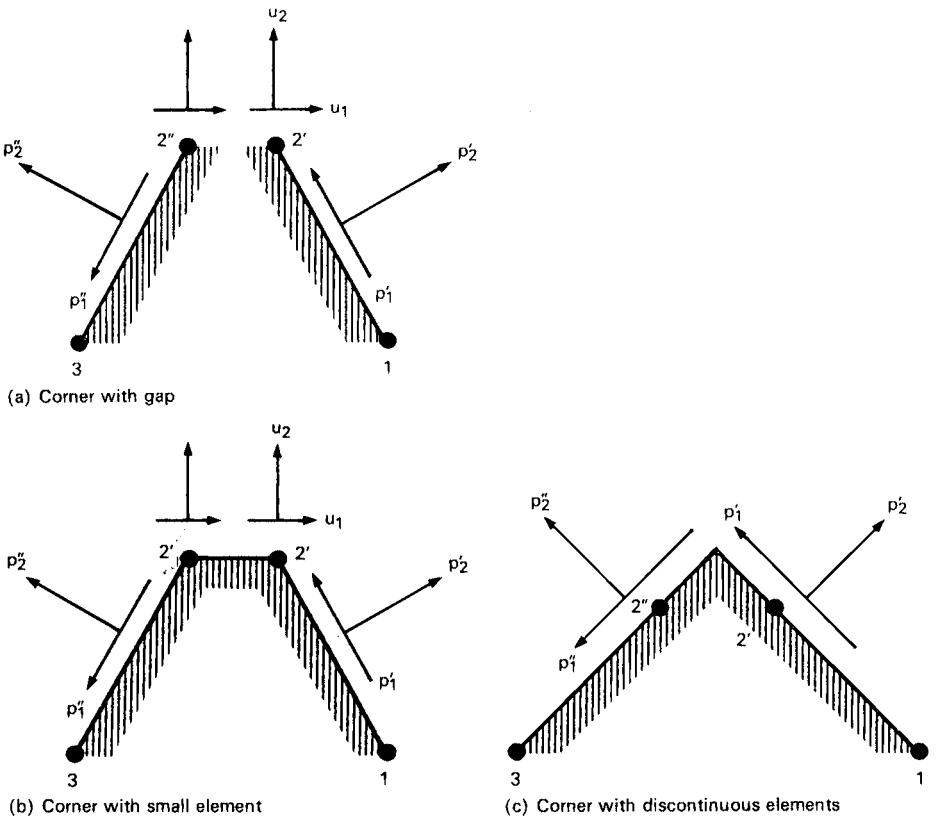

Figure 3.11 Corner tractions and displacements modelling

corner may be computed. This procedure can not be applied for corners where a singularity of the stress tensor occurs.

3.6 Treatment of Domain Integrals and Body Force Terms

Domain integrals are important in boundary elements as they can not only represent body or thermal forces but also a wide range of non-linear effects.

Initial Stress

Many problems produce an initial stress or strain field and in this case one needs to differentiate between the elastic, initial and total components of those variables. If an initial stress state exists, the elastic stresses are given by (see section 3.2)

$$\sigma_{ij} = \sigma^{t}_{ij} + \sigma^{o}_{ij} \tag{3.120}$$

σ^{t}_{ij} represents the 'total' stress and σ^{o}_{ij} the initial one.

If we rewrite the starting equation (3.66), taking into consideration fundamental solution unit forces acting in the 'l' direction, one can write

$$\int_{\Omega} (\sigma_{jk,j} + b_k) u_{lk}^* \, d\Omega = \int_{\Gamma_2} (p_k - \bar{p}_k) u_{lk}^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_{lk}^* \, d\Gamma \tag{3.121}$$

and integrate by parts once, one obtains

$$\int_{\Omega} b_k u_{lk}^* \, d\Omega - \int_{\Omega} \sigma_{jk} \varepsilon_{ljk}^* \, d\Omega = - \int_{\Gamma_2} \bar{p}_k u_{lk}^* \, d\Gamma - \int_{\Gamma_1} p_k u_{lk}^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_{lk}^* \, d\Gamma \tag{3.122}$$

Substituting equation (3.120) one can write,

$$\int_{\Omega} b_k u_{lk}^* \, d\Omega - \int_{\Omega} \sigma_{jk}^{t} \varepsilon_{ljk}^* \, d\Omega - \int_{\Omega} \sigma_{jk}^{\circ} \varepsilon_{ljk}^* \, d\Omega = - \int_{\Gamma_2} \bar{p}_k u_{lk}^* \, d\Gamma - \int_{\Gamma_1} p_k u_{lk}^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_{lk}^* \, d\Gamma \tag{3.123}$$

Integrating by parts again one obtains

$$\int_{\Omega} b_k u_{lk}^* \, d\Omega + \int_{\Omega} \sigma_{ljk,j}^* u_k \, d\Omega - \int_{\Omega} \sigma_{jk}^{\circ} \varepsilon_{ljk}^* \, d\Omega = - \int_{\Gamma_2} \bar{p}_k u_{lk}^* \, d\Gamma - \int_{\Gamma_1} p_k u_{lk}^* \, d\Gamma + \int_{\Gamma_1} \bar{u}_k p_{lk}^* \, d\Gamma + \int_{\Gamma_2} p_{lk}^* \, d\Gamma \tag{3.124}$$

Notice that the integration by parts has been carried out on σ_{jk}^{t} and not on σ_{jk} because the former is the one related to the total displacements. Equation (3.124) gives in more compact form, and after substituting for the fundamental solution, the following expression

$$\int_{\Gamma} u_{lk}^* p_k \, d\Gamma + \int_{\Omega} b_k u_{lk}^* \, d\Omega - \int_{\Omega} \sigma_{jk}^{\circ} \varepsilon_{ljk}^* \, d\Omega = c_{lk}^{i} u_k^{i} + \int_{\Gamma} p_{lk}^* u_k \, d\Gamma \tag{3.125}$$

This demonstrates that the initial stress (and similarly for strain) field can be treated in a similar way as a body force field, b_k, although it is generally difficult to transform the σ° term into a boundary integral.

Body Forces

Let us study in this section how the body force term in (3.125) can be taken to the boundary. The integral under study is

$$\int_{\Omega} b_k u_{lk}^* \, d\Omega \tag{3.126}$$

The integral does not include any unknown values but in order to compute it numerically the whole domain has to be discretized into cells which represents a

considerable difficulty and diminishes the elegance and computational efficiency of the method. Fortunately domain integrals can be avoided in many practical cases by reducing them to the boundary. Consider in this case that the body force components b_k can be obtained from a potential function ψ such that,

$$b_k = \frac{\partial \psi}{\partial x_k} \tag{3.127}$$

where the potential ψ is assumed to satisfy the following harmonic relationship

$$\nabla^2 \psi = K_0 = \text{constant} \tag{3.128}$$

Then integral (3.126) can be rewritten as

$$\begin{aligned} \int_\Omega u_{lk}^* b_k \, d\Omega &= \int_\Omega u_{lk}^* \left(\frac{\partial \psi}{\partial x_k} \right) d\Omega \\ &= \int_\Omega \frac{\partial}{\partial x_k} (u_{lk}^* \psi) \, d\Omega - \int_\Omega \frac{\partial u_{lk}^*}{\partial x_k} \psi \, d\Omega \end{aligned} \tag{3.129}$$

where derivatives by parts have been used. For the divergence theorem one can now write

$$\int_\Omega u_{lk}^* b_k \, d\Omega = \int_\Gamma u_{lk}^* \psi n_k \, d\Gamma - \int_\Omega \frac{\partial u_{lk}^*}{\partial x_k} \psi \, d\Omega \tag{3.130}$$

where n_k is the direction cosine of the normal to Γ with respect to x_k axis. In order to take the last integral to the boundary one can write the fundamental solution u_{lk}^* in terms of its Galerkin vector G_{lk} in the form

$$u_{lk}^* = G_{lk,jj} - \frac{1}{2(1-\nu)} G_{lj,kj} = \nabla^2 (G_{lk}) - \frac{1}{2(1-\nu)} G_{lj,kj} \tag{3.131}$$

where the comma indicates derivatives.

The Galerkin vector for the fundamental solution of three dimensional elastostatics problems was given in section 3.3 and is repeated below, i.e.

$$G_{kl} = \frac{1}{8\pi\mu} r \delta_{kl} \tag{3.132}$$

For two dimensional problems,

$$G_{kl} = \frac{1}{8\pi\mu} r^2 \ln\left(\frac{1}{r}\right) \delta_{kl} \tag{3.133}$$

It must be pointed out that when the two-dimensional Galerkin vector is derived, the following fundamental solution is obtained,

$$u_{lk}^{*} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\frac{1}{r}\,\delta_{kl} - \left(\frac{7-8\nu}{2}\right)\delta_{kl} + r_{,k}r_{,l}\right] \tag{3.134}$$

This solution differs from the previous fundamental solution (equation (3.54)) by a constant term, which was dropped before because a rigid body motion does not change the solution of the systems of equations. However, when body forces exist and the Galerkin vector given by (3.133) is used to take the domain integral to the boundary, the fundamental solution given by (3.134) must be used.

By applying (3.131) one can write the last integral in (3.130) as follows,

$$\int_{\Omega} u_{lk,k}^{*}\psi\, d\Omega = \frac{1-2\nu}{2(1-\nu)}\int_{\Omega}\nabla^2(G_{lk,k})\psi\, d\Omega \tag{3.135}$$

Finally, the Green's theorem can be applied between the field $G_{lk,k}$ and ψ, i.e.

$$\int_{\Omega}(G_{lk,k}\nabla^2\psi - \nabla^2(G_{lk,k})\psi)\, d\Omega$$

$$= \int_{\Gamma}(G_{lk,k}\psi_{,j}n_j)\, d\Gamma - \int_{\Gamma}(G_{lk,kj}\psi n_j)\, d\Gamma \tag{3.136}$$

Notice that the first left hand side integral in (3.136) is simply

$$\int_{\Omega} G_{lk,k}\nabla^2\psi\, d\Omega = K_0\int_{\Omega} G_{lk,k}\, d\Omega = K_0\int_{\Gamma} G_{lk}n_k\, d\Gamma \tag{3.137}$$

Here, equation (3.136) can be written as,

$$-\int_{\Omega}\nabla^2(G_{lk,k})\psi\, d\Omega = \int_{\Gamma}(G_{lk,k}\psi_{,j}n_j)\, d\Gamma$$

$$-\int_{\Gamma}(G_{lk,kj}\psi n_j)\, d\Gamma - K_0\int_{\Gamma} G_{lk}n_k\, d\Gamma \tag{3.138}$$

One can now substitute equation (3.138) into (3.135) and this into (3.130) to obtain the expression for the body force integral in terms of boundary integrals, i.e.

$$\int_{\Omega} u_{lk}^{*}b_k\, d\Omega = \int_{\Gamma} u_{lk}^{*}\psi n_k\, d\Gamma + \frac{1-2\nu}{2(1-\nu)}$$

$$\times\left\{\int_{\Gamma}(G_{lk,k}\psi_{,j}n_j)\, d\Gamma - \int_{\Gamma}(G_{lk,kj}\psi n_j)\, d\Gamma - K_0\int_{\Gamma} G_{lk}n_k\, d\Gamma\right\} \tag{3.139}$$

We will now specialize equation (3.139) for different types of loads.

Gravitational Loads

As an example of body forces, the case of gravitational loads will be considered first as they are simple to express, i.e.

$$\psi = -\rho g x_3; \qquad \mathbf{b} = \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix}; \qquad \nabla^2 \psi = 0 \tag{3.140}$$

where ρ is the density and g the gravitational acceleration.

The domain integral for this case can be written as

$$\int_\Omega u^*_{lk} b_k \, d\Omega = -\int_\Gamma u^*_{lk} \rho g x_3 n_k \, d\Gamma - \frac{(1-2\nu)\rho g}{16\pi\mu(1-\nu)} \int_\Gamma (r_{,l} n_3 - x_3 r_{,lj} n_j) \, d\Gamma \tag{3.141}$$

Notice that in this case the internal stress σ_{ij} will also have a body force term which can also be taken to the boundary using the above method.

Thermoelastic Problems

The temperature changes θ in an elastic body are equivalent to adding a body force equal to $(-\gamma\theta_{,k})$ at each point and increasing the tractions by $(\gamma\theta n_k)$ where

$$\gamma = \frac{2\mu\alpha(1+\nu)}{(1-2\nu)} \tag{3.142}$$

α is the coefficient of thermal expansion. Hence the thermoelastic problem is a particular case of the elastostatic problem with body forces.

The boundary integral equation for a thermoelastic body without any other types of body forces is

$$c_{lk} u^i_k + \int_\Gamma p^*_{lk} u_k \, d\Gamma = \int_\Gamma u^*_{lk} p_k \, d\Gamma + \int_\Gamma u^*_{lk} (\gamma\theta) n_k \, d\Gamma - \int_\Omega u^*_{lk} \gamma\theta_{,k} \, d\Omega \tag{3.143}$$

Comparing to (3.139) we can write the potential of the equivalent body forces as,

$$\psi = -\gamma\theta \tag{3.144}$$

Taking into account that for steady state conduction one has $\nabla^2\theta \equiv 0$ the domain integral in (3.143) then becomes a boundary integral and (3.143) can be written as

$$c_{lk} u^i_k + \int_\Gamma p^*_{lk} u_k \, d\Gamma = \int_\Gamma u^*_{lk} p_k \, d\Gamma + \frac{(1-2\nu)\gamma}{2(1-\nu)} \int_\Gamma (G_{lk,kj} \theta n_j - G_{lk,k} \theta_{,j} n_j) \, d\Gamma \tag{3.145}$$

The above integral equation may be discretized in the usual way. All the functions in the last integral of equation (3.145) are known and may be easily integrated by numerical procedures over the boundary elements on the surface.

Thermoelastic problems may also be studied considering the temperature effects as initial strains, i.e

$$\varepsilon^{\circ}_{jk} = \alpha\theta\delta_{jk} \tag{3.146}$$

with the initial stresses given by

$$\sigma^{\circ}_{jk} = -(\lambda\varepsilon^{\circ}_{ii}\delta_{jk} + 2\mu\varepsilon^{\circ}_{jk}) \tag{3.147}$$

The values of σ°_{jk} are

$$\sigma^{\circ}_{jk} = -2\mu\left(\frac{1+\nu}{1-2\nu}\right)\alpha\theta\delta_{jk} = -\gamma\theta\delta_{jk} \tag{3.148}$$

Substituting equation (3.148) into the initial stress term of the elastic equation (3.125) one obtains,

$$-\int_{\Omega}\sigma^{\circ}_{jk}\varepsilon^{*}_{ljk}\,d\Omega = \int_{\Omega}\gamma\theta\delta_{jk}\varepsilon^{*}_{ljk}\,d\Omega = \int_{\Omega}\gamma\theta u^{*}_{lk,k}\,d\Omega$$

$$= -\int_{\Omega}\gamma\theta_{,k}u^{*}_{lk}\,d\Omega + \int_{\Gamma}\gamma\theta u^{*}_{lk}n_k\,d\Gamma \tag{3.149}$$

When equation (3.149) is substituted into (3.125) without any other type of body force, one obtains once more equation (3.143) that after taking the domain integral to the boundary gives (3.145). As could be expected both interpretations of thermoelastic effects give identical results.

3.7 Subregions in Elasticity

In the previous sections the boundary elements for homogeneous linear elastic isotropic media have been formulated. In many cases media that are not homogeneous but consist of several zones each one being homogeneous must be analysed. Those problems may be studied using the above formulation of the BEM and continuity conditions. All the boundaries of the body have to be discretized, including internal boundaries that separate homogeneous zones within the medium. The equations formulated for every homogeneous zone plus the displacement and traction continuity conditions over the internal boundaries produce a system that may be solved once the external boundary conditions are taken into account.

Consider the problem shown in figure 3.12 consisting of three zones of different elastic materials. A two-dimensional domain has been represented for simplicity; however, the procedure described below applies to both two and three-dimensional problems.

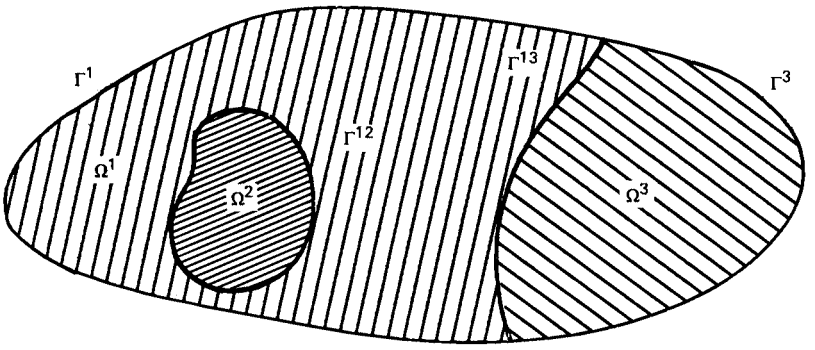

Figure 3.12 Zoned media

The following symbols are used:

Γ^i : external part of the boundary of zone Ω^i
Γ^{ij} : boundary between zones Ω^i and Ω^j
$\mathbf{u}^i, \mathbf{p}^i$: nodal displacements and tractions at nodes on the boundary Γ^i of zone Ω^i
$\mathbf{u}^{ij}, \mathbf{p}^{ij}$: nodal displacements and tractions at nodes on Γ^{ij} as part of zone Ω^i
$\mathbf{H}^i, \mathbf{G}^i$: parts of the H and G matrices obtained for zone Ω^i that multiply $\mathbf{u}^i$ and $\mathbf{p}^i$, respectively
$\mathbf{H}^{ij}, \mathbf{G}^{ij}$: parts of the H and G matrices obtained for zone Ω^i that multiply $\mathbf{u}^{ij}$ and $\mathbf{p}^{ij}$, respectively.

The BEM equations for the three homogeneous zones of figure 3.12 are

$$[\mathbf{H}^1 \quad \mathbf{H}^{12} \quad \mathbf{H}^{13}] \begin{Bmatrix} \mathbf{u}^1 \\ \mathbf{u}^{12} \\ \mathbf{u}^{13} \end{Bmatrix} = [\mathbf{G}^1 \quad \mathbf{G}^{12} \quad \mathbf{G}^{13}] \begin{Bmatrix} \mathbf{p}^1 \\ \mathbf{p}^{12} \\ \mathbf{p}^{13} \end{Bmatrix} \tag{3.150}$$

for zone Ω^1

$$\mathbf{H}^{21} \quad \mathbf{u}^{21} = \mathbf{G}^{21} \quad \mathbf{p}^{21} \tag{3.151}$$

for zone Ω^2 and

$$[\mathbf{H}^3 \quad \mathbf{H}^{31}] \begin{Bmatrix} \mathbf{u}^3 \\ \mathbf{u}^{31} \end{Bmatrix} = [\mathbf{G}^3 \quad \mathbf{G}^{31}] \begin{Bmatrix} \mathbf{p}^3 \\ \mathbf{p}^{31} \end{Bmatrix} \tag{3.152}$$

for zone Ω^3. The traction equilibrium conditions and displacement compatibility

conditions over the internal boundaries Γ^{ij} are

$$\begin{aligned} \mathbf{u}^{12} &= \mathbf{u}^{21} \\ \mathbf{u}^{13} &= \mathbf{u}^{31} \\ \mathbf{p}^{12} &= -\mathbf{p}^{21} \\ \mathbf{p}^{13} &= -\mathbf{p}^{31} \end{aligned} \tag{3.153}$$

that transform equation (3.151) and (3.152) into

$$[\mathbf{H}^{21} \quad \mathbf{G}^{21}] \begin{Bmatrix} \mathbf{u}^{12} \\ \mathbf{p}^{12} \end{Bmatrix} = 0 \tag{3.154}$$

$$[\mathbf{H}^{3} \quad \mathbf{H}^{31} \quad \mathbf{G}^{31}] \begin{Bmatrix} \mathbf{u}^{3} \\ \mathbf{u}^{13} \\ \mathbf{p}^{13} \end{Bmatrix} = \mathbf{G}^{3}\mathbf{p}^{3} \tag{3.155}$$

The last two equations plus equation (3.150) can be rearranged into

$$\begin{bmatrix} \mathbf{H}^{1} & \mathbf{0} & \mathbf{H}^{12} & -\mathbf{G}^{12} & \mathbf{H}^{13} & -\mathbf{G}^{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}^{21} & \mathbf{G}^{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{3} & \mathbf{0} & \mathbf{0} & \mathbf{H}^{31} & \mathbf{G}^{31} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{1} \\ \mathbf{u}^{3} \\ \mathbf{u}^{12} \\ \mathbf{p}^{12} \\ \mathbf{u}^{13} \\ \mathbf{p}^{13} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^{3} \end{bmatrix} \begin{Bmatrix} \mathbf{p}^{1} \\ \mathbf{p}^{3} \end{Bmatrix} \tag{3.156}$$

The above system of equations may be solved once the boundary conditions on Γ^1 and Γ^3 are prescribed. The total number of unknowns is equal to the number of nodal degrees of freedom over the external boundaries plus twice the number of nodal degrees of freedom over the internal boundaries.

The subdivision of the region into several zones may be used also in homogeneous media as a way of avoiding numerical problems or improving computational efficiency. For instance problems that include cracks or notches, as the one shown in figure 3.13, present numerical difficulties when the boundary is discretized due to the proximity of some of the nodes. The difficulties disappear if the region is divided into two zones in a way that the nodes that are very close belong to different regions. Another situation where the subdivision of a homogeneous region may be useful corresponds to problems with a large number of unknowns, as the one shown in figure 3.14. In those cases the subdivision transforms the fully populated matrices into banded matrices, which are more convenient from a computational point of view. In those cases, the increase in the number of unknowns because of the internal boundaries must be small, otherwise the subdivision will be worthless.

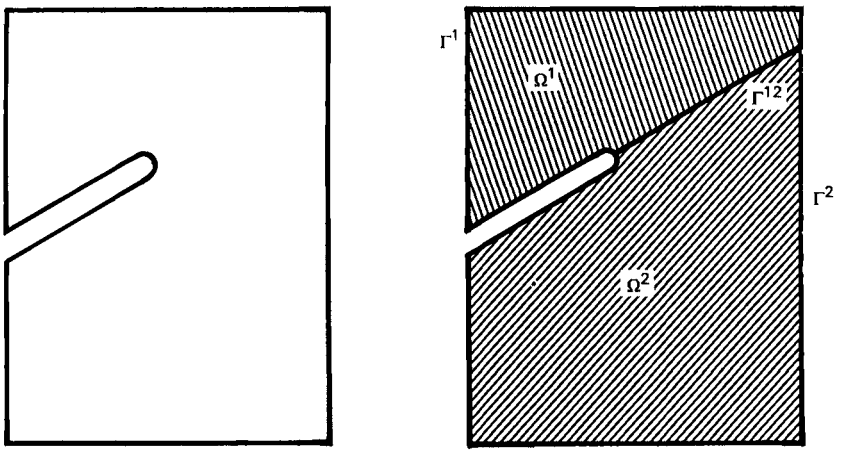

Figure 3.13 Subdivision to avoid numerical difficulties in a problem with a notch

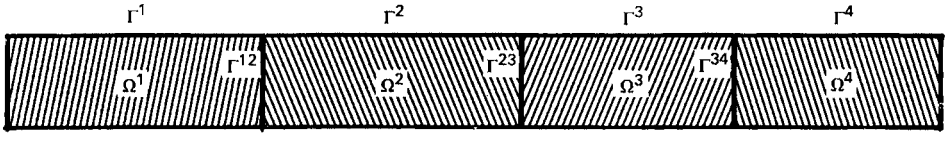

Figure 3.14 Subdivision to increase computational efficiency

3.8 Indirect Formulations

The fundamental boundary integral equation used throughout this chapter is known as the direct integral representation and gives displacements at internal and boundary points in terms of boundary tractions and boundary displacements. There are some other integral representations where the displacements are written in function of variables which are not explicitly the boundary displacements or tractions. Those representations are the basis of the so-called indirect boundary element formulations.

Let us consider a state σ_{ij}, ε_{ij}, u_j, p_j over the domain Ω of figure 3.15(a). We can now define another state over the complementary domain Ω' (figure 3.15(b)) with variables σ'_{ij}, ε'_{ij}, u'_j, p'_j. The tractions over the external region of Ω' will be referred to the normal n of the internal domain as shown in the figure.

It is easy to establish a reciprocity relationship for the complementary domain between the fundamental solution applied at a point in Ω and the complementary state indicated by primes. This gives,

$$\int_\Gamma u^*_{ik} p'_k \, d\Gamma - \int_\Gamma p^*_{ik} u'_k \, d\Gamma = 0 \tag{3.157}$$

This relationship can be subtracted from the integral equation for the reference state, as given by (3.71), i.e.

$$u_l^i - \int_\Gamma u_{lk}^* p_k \, d\Gamma + \int_\Gamma p_{lk}^* u_k \, d\Gamma = 0 \tag{3.158}$$

Notice that the body forces are not included for simplicity.

Subtracting (3.157) from (3.158) one obtains

$$u_l^i = \int_\Gamma u_{lk}^* (p_k - p_k') \, d\Gamma + \int_\Gamma p_{lk}^* (u_k' - u_k) \, d\Gamma \tag{3.159}$$

Since the complementary state can be defined arbitrarily, it will be assumed that it is such that its displacements on the boundary are the same as those of the original solution, i.e.

$$u_k' = u_k \qquad \text{on } \Gamma \tag{3.160}$$

Calling $\sigma_k = p_k - p_k'$ equation (3.159) can be rewritten as

$$u_l^i = \int_\Gamma u_{lk}^* \sigma_k \, d\Gamma \tag{3.161}$$

This equation can be interpreted as the displacement at a point 'i' inside Ω and can be obtained by the summation of displacements due to loads $\sigma_k \, d\Gamma$ applied at every $d\Gamma$ when Ω is considered to be part of the complete region. The integral representation in equation (3.161) is known as the single layer potential with density σ_k. As can be seen from equation (3.160) the displacements are considered to be continuous on the boundary while the tractions are discontinuous.

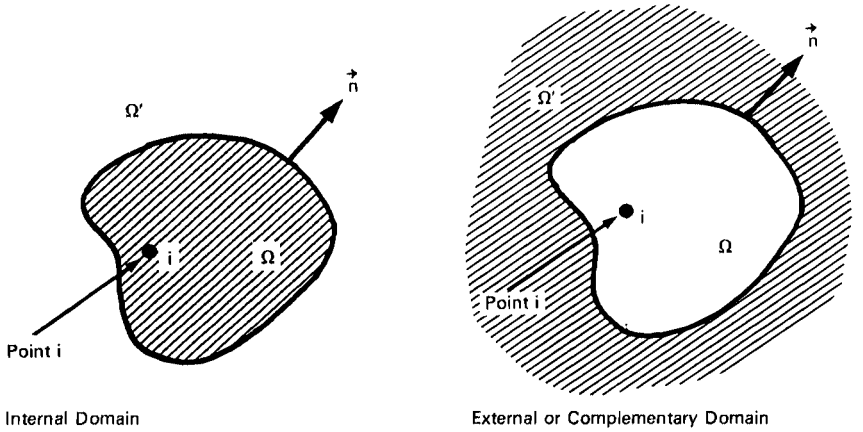

Figure 3.15 Internal and external domains

Tractions at any point can then be computed by carrying out the derivatives of (3.161) and in this case give

$$p_l = \int_\Gamma p^*_{lk} \sigma_k \, d\Gamma \tag{3.162}$$

Another possibility is to assume that the complementary state is such that

$$p'_j = p_j \qquad \text{on } \Gamma \tag{3.163}$$

Calling $\omega_k = u'_k - u_k$ equation (3.159) can be written as,

$$u^i_l = \int_\Gamma p^*_{lk} \omega_k \, d\Gamma \tag{3.164}$$

The above integral representation is known as double layer potential with intensity ω_k. As can be seen by the definition of ω, the double layer approach produces displacements which are discontinuous on γ while the tractions are continuous. Equation (3.164) can also be physically interpreted as a superposition of the displacements at i when dislocations $\omega_k \, d\Gamma$ are applied at every $d\Gamma$, with Ω considered to be part of the complete region. Tractions are calculated by carrying out derivatives in (3.164) which has the disadvantage over the previous indirect formulation of producing a higher order singularity.

It has been shown that internal displacements can be represented by a single layer potential (equation (3.161)), a double layer potential (equation (3.164)) or a combination of both, which is the basis of the direct formulation.

In general, the internal stresses and displacements computed by means of the discretized form of the integral representations are more accurate than those obtained using other numerical methods and similar discretizations. This is a consequence of the fact that the internal values are obtained by integration of fundamental solutions that are exact and only the boundary densities of the potentials are approximated. According to St. Venant's principle the local errors of this approximation may be expected to damp quickly.

The method based on the solution of the above integral equations by boundary discretization are known as Indirect Boundary Element Methods. Sometimes this name is used only for the single layer potential representation while the method based on the double layer potential is called the Displacement Discontinuity Method.

Example 3.3 Foundations

One of the most important aspects of the design of foundations is the computation of their stiffness or impedance. This is represented by the matrix **K** which relates applied loads to displacements,

$$\mathbf{F} = \mathbf{K}\mathbf{U} \tag{(a)}$$

Although here we will only refer to the static stiffness of foundations, it is important to point out that their dynamic response may be important in many other applications. This problem can also be solved using boundary elements but its discussion is beyond the scope of this text.

Figure 3.16 shows a transversal section of a foundation embedded in the soil. In many cases the behaviour of the soil can be assumed to be isotropic and linear and represented by a half-space. As the stiffness of the foundation is much larger this is assumed to behave as a rigid body and moments or forces applied at the top to simulate the loadings. When applying the Kelvin fundamental solution to the three dimensional soil domain, one needs to discretize not only the soil-foundation interface but also the free surface of the soil. This usually introduces an approximation because the discretization of the free surface is only carried out up to a certain distance from the foundations or because one develops an approximate infinite element. Figure 3.17 shows such a discretization for a square embedded foundation with level of embedment $E/B = 4/3$ and an amount of free field defined by $A/B = 2.5$. The analysis was carried out using constant elements and the axis of rotation for the rocking motion was considered to be on the soil-foundation interface. Over the free surface elements the tractions were considered to be zero.

The variation of stiffness with number of elements along half of the side of the bottom of the embedded foundation is shown in figure 3.18 (the discretization of the lateral walls is varied consistently). The figure shows that the boundary element mesh does not need to be very refined to obtain accurate results, mainly because in this case we are interested in the integrated tractions – i.e. resultant forces and moments – rather than the stresses along the foundation. Results for $N = 6$ compared favourably with those obtained with finer meshes.

The effect of the amount of soil free surface that is discretized can be seen in figure 3.19 for the same foundation. The study has been carried out adding successively lines of constant elements and the previous discretization of the soil free surface. Results converge rapidly.

The discretizaton of the soil free surface may be avoided by using Mindlin's fundamental solution [9] instead of Kelvin. This solution corresponds to the point load in an elastic half-space and reduces the number of elements required to run this type of problem. The computer time per integration over a boundary element

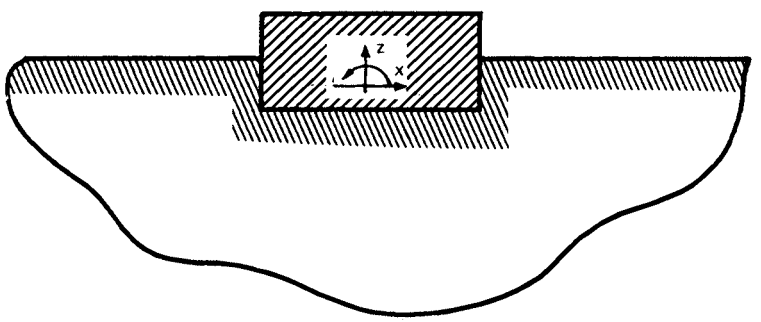

Figure 3.16 Rigid foundation embedded in the soil

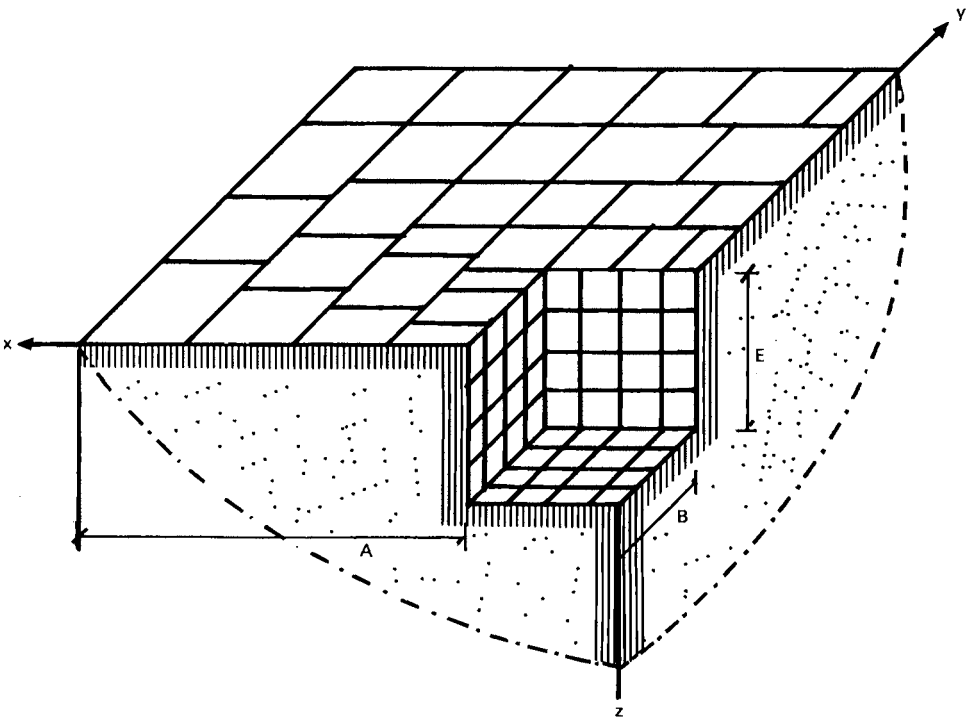

Figure 3.17 Discretization for one quarter of a square embedded foundation

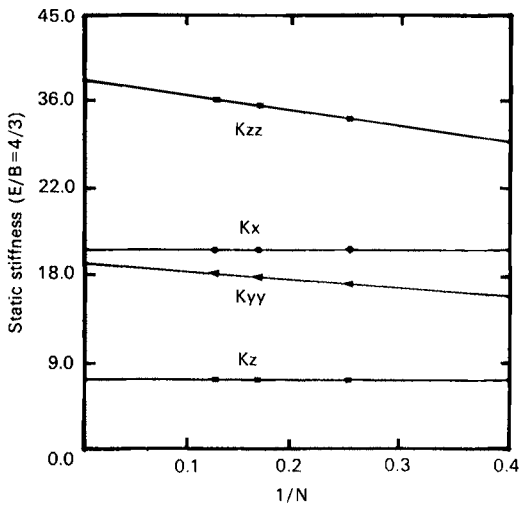

Figure 3.18 Effect of the interface discretization on the foundation stiffness. (Results are normalized as follows: Horizontal stiffness: $K_{x\,norm} = K_x(2-\nu)/GB$; Vertical stiffness: $K_{z\,norm} = K_z(1-\nu)/GB$; Rocking stiffness: $K_{yy\,norm} = K_{yy}(1-\nu)/GB^3$; and Torsional stiffness: $K_{zz\,norm} = K_{zz}/GB^3$.)

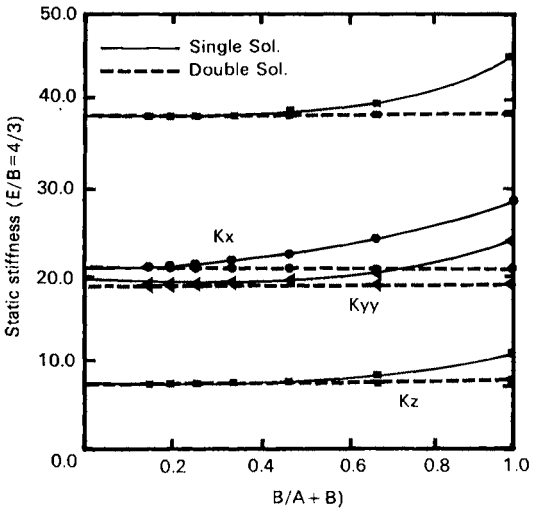

Figure 3.19 Effect of the free surface discretization on the foundation stiffnesses

grows however due to the greater amount of terms involved in the fundamental solution. Another computationally more efficient approach is based on the use of Kelvin's solution in conjunction with another point load solution following the same direction but located at the image point of the first with respect to the soil surface. The use of this solution reduces to zero the surface tractions with the exception of p_{13}, p_{23} and p_{31}, p_{32} which are expected to have a small effect in soils. Figure 3.19 shows how when the double solution is used the result becomes almost independent of the amount of free surface discretized and the free surface does not have to be considered.

Many other soil problems can be studied using the boundary element method. Some applications in zoned media are shown in figure 3.20, where it can be seen that the soil can be inhomogeneous.

3.9 Axisymmetric Problems

There are many elastic problems that present an axisymmetric geometry and very frequently also axisymmetric loading conditions. In the following, it is first explained how boundary elements may be applied to axisymmetric problems with respect to both the geometry and the loading by taking advantage of the symmetry that reduces a three-dimensional analysis to two uncoupled plane domain problems; one with two degrees of freedom per point (radial and axial) and another with one degree of freedom per point (tangential). Non-axisymmetric loads are studied later. The first boundary elements formulation for axisymmetric elastic problems was published in 1975 [11], [12]. It was based on the Somigliana's identity obtained from the application of the reciprocity theorem between the actual axisymmetric problem and the fundamental solutions corresponding to a radial

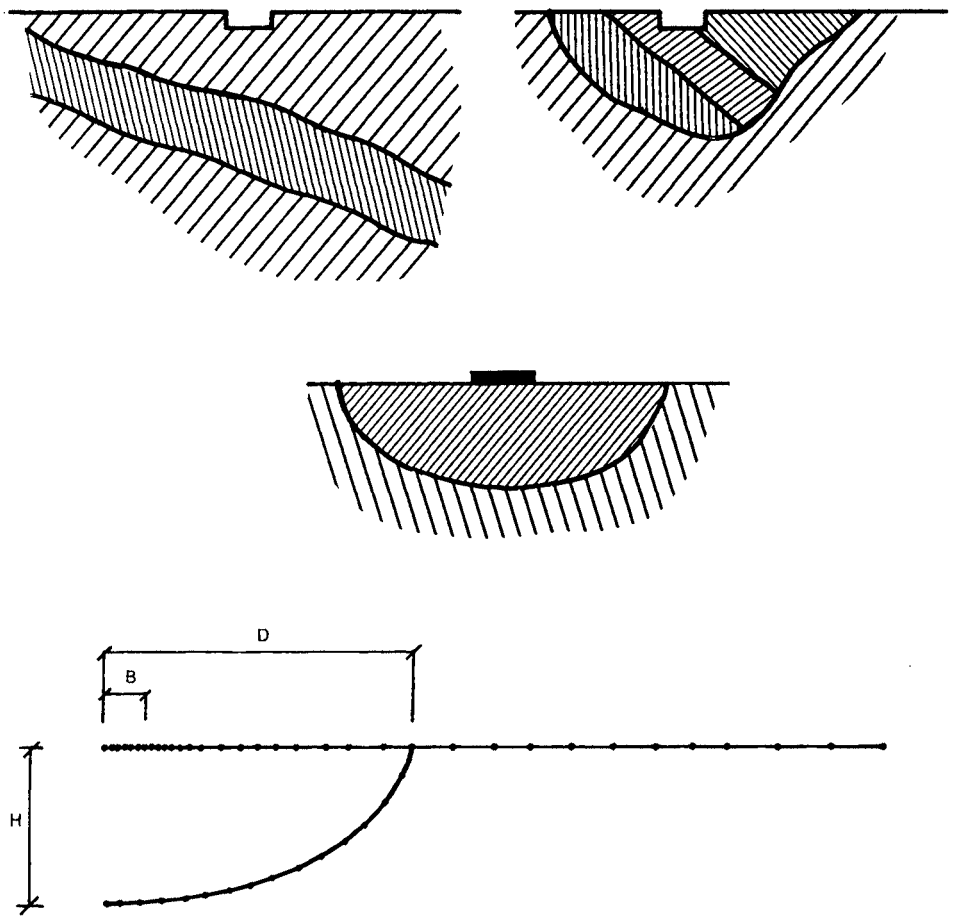

Figure 3.20 Zoned soil models and boundary discretization

ring load and an axial ring load, for one part of the problem, and a tangential ring load for the other (torsion) part of the problem, as indicated in figure 3.21.

The basic equation has now the same expression as in previous cases (equation (3.87)) with $\mathbf{u}^*$ and $\mathbf{p}^*$ being the displacements and tractions due to the ring loads. Those fundamental solutions were obtained by different procedures by Kermanidis [11], Mayr [12], Cruse *et al.* [13], and Dominguez and Abascal [9]. They are written in terms of Legendre functions or complete elliptic integrals, which makes their integration along the boundary elements rather involved. Explicit expressions of the ring loads fundamentals solutions may be found, for instance, in [11], [13]. The procedure is basically the same as that presented in section 2.15 for axisymmetric potential problems. An alternative and more general approach based on the three dimensional formulation is presented in what follows.

The three dimensional fundamental solution will be used and numerical integration of $\mathbf{p}^*$ and $\mathbf{u}^*$ performed on the axisymmetric elements.

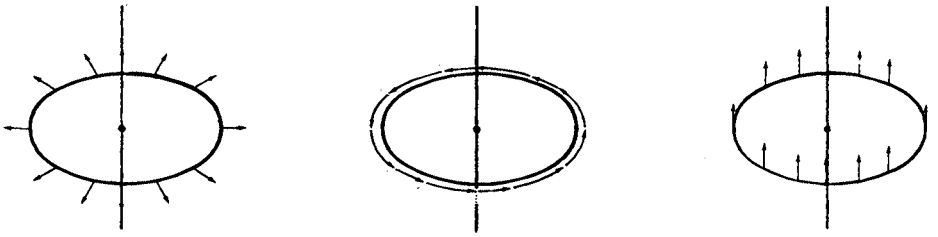

Figure 3.21 Ring loads

The basic boundary elements equation for zero body forces can be written in cylindrical coordinates using matrix notation as,

$$\mathbf{c}_c^i \mathbf{u}_c^i + \int_\Gamma \mathbf{p}_c^* \mathbf{u}_c \, d\Gamma = \int_\Gamma \mathbf{u}_c^* \mathbf{p}_c \, d\Gamma \tag{3.165}$$

where the subscript c stands for cylindrical.

The relation between cartesian and cylindrical coordinates may be written for vectors $\mathbf{u}$ and $\mathbf{p}$ as

$$\begin{aligned} \mathbf{u}_c &= \mathbf{Q}^T \mathbf{u} \\ \mathbf{p}_c &= \mathbf{Q}^T \mathbf{p} \end{aligned} \tag{3.166}$$

for the matrix $\mathbf{c}^i$

$$\mathbf{c}_c^i = \mathbf{Q}^{i,T} \mathbf{c}^i \mathbf{Q}^i \tag{3.167}$$

and for matrices $\mathbf{p}^*$ and $\mathbf{u}^*$

$$\begin{aligned} \mathbf{u}_c^* &= \mathbf{Q}^{i,T} \mathbf{u}^* \mathbf{Q} \\ \mathbf{p}_c^* &= \mathbf{Q}^{i,T} \mathbf{p}^* \mathbf{Q} \end{aligned} \tag{3.168}$$

where (figure 3.22)

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.169}$$

Notice that $\mathbf{u}^*$ and $\mathbf{p}^*$ relate the collocation point i and the integration point.

Since i is the collocation point, it may be assumed that $\theta^i = 0$ (figure 3.22), which makes the transformation matrix $\mathbf{Q}^i = \mathbf{I}$. The kernels of the integrals in

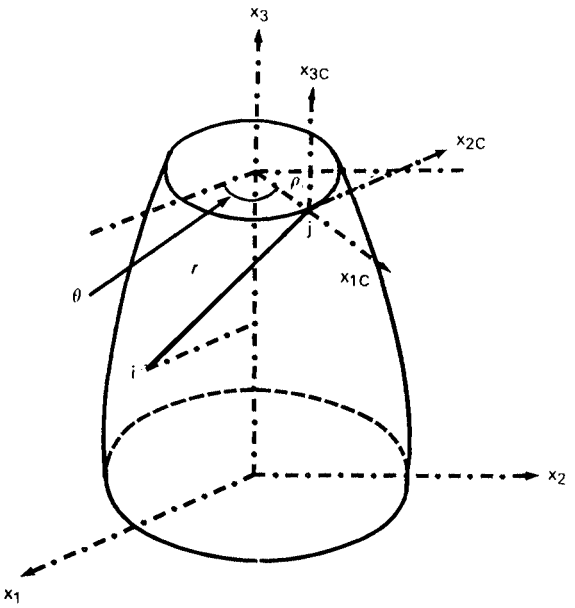

Figure 3.22 Cylindrical coordinates

equation (3.165) have the form

$$\mathbf{u}_c^* = \mathbf{u}^*\mathbf{Q} = \begin{bmatrix} u_{11}^* \cos\theta + u_{12}^* \sin\theta & -u_{11}^* \sin\theta + u_{12}^* \cos\theta & u_{13}^* \\ u_{21}^* \cos\theta + u_{22}^* \sin\theta & -u_{21}^* \sin\theta + u_{22}^* \cos\theta & u_{23}^* \\ u_{31}^* \cos\theta + u_{32}^* \sin\theta & -u_{31}^* \sin\theta + u_{32}^* \cos\theta & u_{33}^* \end{bmatrix} \tag{3.170}$$

One half of a meridional section of the body is discretized into elements (figure 3.23) and equation (3.165) may be written for the boundary node i as

$$\mathbf{c}^i\mathbf{u}^i + \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} \rho \left[\int_0^{2\pi} \mathbf{p}_c^* \, d\theta \right] \boldsymbol{\phi} \, d\Gamma \right\} \mathbf{u}^j = \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} \rho \left[\int_0^{2\pi} u_c^* \, d\theta \right] \boldsymbol{\phi} \, d\Gamma \right\} \mathbf{p}^j \tag{3.171}$$

where ρ is the radius shown in figure 3.22, Γ_j are the boundary elements, $\boldsymbol{\phi}$ the usual shape functions for two dimensional problems and the subindex c has been dropped out for simplicity.

The integration along θ may be easily done by numerical procedures. The submatrices $\mathbf{H}^{ij}$ and $\mathbf{G}^{ij}$ that relate two nodes i and j have the pattern

$$\begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix} \begin{matrix} \leftarrow \rho \\ \leftarrow \theta \\ \leftarrow z \end{matrix} \tag{3.172}$$

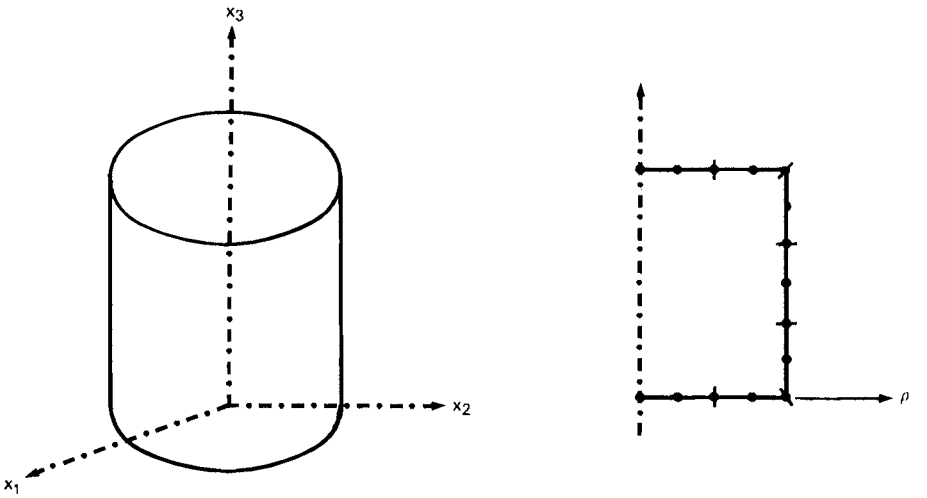

Figure 3.23 Quadratic BE discretization for axisymmetric problem

where the zeros denote elements that are null due to the skewsymmetry of the corresponding terms in equation (3.170). It is clear that the torsion and the radial-axial problems are uncoupled and both may be studied on a plane domain.

A special numerical integration scheme for axisymmetric problems was proposed by Gomez-Lera *et al.* [14] for axisymmetric problems using the three-dimensional point load fundamental solution. When Kelvin's solution is integrated around the axis, a Gauss quadrature may be applied to every semi-ring; however, its accuracy is easily improved by increasing the density of integration points near the collocation points by means of a parabolic transformation of the circumferential coordinate

$$\theta = \frac{\pi}{4}(\xi + 1)^2; \qquad -1 \leqslant \xi \leqslant 1 \tag{3.173}$$

It should be noticed that since a rigid body motion along the radial coordinate of an axisymmetric body is meaningless, the corresponding terms of $\mathbf{c}^i$ have to be computed from the rest of the terms of the **G** and **H** matrices giving a uniform shrinking of the ξ coordinate or using the analytical expression given for plane problems in the next chapter.

When the boundary conditions are not axisymmetric, the problem may still be analysed by means of a plane model. The problem is divided into a number of uncoupled plane problems by representing the prescribed loading or displacement by a Fourier series along the tangential coordinate [15]. Each term of the series produces displacements and stresses in the same Fourier mode and if the prescribed values do not vary very rapidly around the axis, a few modes will be enough for an accurate solution. The Fourier expansion is of the form

$$u_\rho = \sum_{n=0}^{\infty} (u^s_{n\rho} \cos n\theta + u^a_{n\rho} \sin n\theta)$$

$$u_\theta = \sum_{n=0}^{\infty} (-u^s_{n\theta} \sin n\theta + u^a_{n\theta} \cos n\theta) \tag{3.174}$$

$$u_z = \sum_{n=0}^{\infty} (u^s_{nz} \cos n\theta + u^a_{nz} \sin n\theta)$$

where 's' indicates the symmetric terms and 'a' the antisymmetric ones.

For each Fourier mode amplitude a discretized boundary equation like equation (3.171) may be written with the only difference being that $\mathbf{p}_c^*$ and $\mathbf{u}_c^*$ are now weighted by a sine or a cosine function and the integrals around the axis are of the form

$$\int_0^{2\pi} \mathbf{u}_c^* \sin n\theta \, d\theta; \qquad \int_0^{2\pi} \mathbf{u}_c^* \cos n\theta \, d\theta \tag{3.175}$$

It is worth noting that since $\sin n\theta$ has zero value at $\theta = 0$, the i point cannot be located at $\theta^i = 0$ to compute the amplitude of those terms of the Fourier series. One only has to move i to a point, for instance $\theta^i = -\pi/2n$,where the amplitude is not zero. This change is easily taken into account by a shift of the origin of θ in equation (3.170).

3.10 Anisotropic Elasticity

The constitutive relationships described in section 3.2 are valid for isotropic cases for which the behaviour of the material can be described in terms of only two constants. In more general cases the material can be orthotropic or generaly anisotropic. For a three dimensional anisotropic case the number of different elastic constants is 21 and they can be expressed in matrix form as,

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \tag{3.176}$$

where $\mathbf{D}$ is a 6×6 matrix. The $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the usual six stress and strain components. The above elastic relationship can also be written in index form which gives a notation more consistent with the previous sections, i.e.

$$\sigma_{ij} = d_{ijkl}\varepsilon_{kl} \tag{3.177}$$

d_{ijkl} are called the rigidity coefficients. The inverse of (3.176) produces the elastic compliance, c_{ijkl}, i.e.

$$\varepsilon_{kij} = c_{ijkl}\sigma_{kl} \tag{3.178}$$

Although there are in general 21 constants for an elastic material, their number is reduced when the material structure has one or more planes of symmetry.

Materials with the orthogonal planes of symmetry are said to be orthotropic and the number of independent coefficients reduces to 9.

For two dimensional problems, the plane $x_1 - x_2$ is a plane of symmetry for the material structure. In this case the number of independent coefficients reduces to 13. Furthermore if either the x_1 or x_2 are axes of symmetry, the material is orthotropic and the constants reduce to 9 as before.

Fundamental solutions exist for the case of two and three dimensional anisotropic cases but they are difficult to use because of the complexity of their mathematical formulation or the need to find part of the solution numerically, which may be inefficient.

Because of this a different technique will be described in this section which can be applied to any anisotropic material. The approach consists of using a reference isotropic fundamental solution and iterating to find the correct result. The procedure will be explained in what follows.

Consider again the starting weighted residual statement, in terms of the equilibrium equation (3.66), i.e.

$$\int_{\Omega} (\sigma_{kj,j} + b_k) u_k^* \, d\Omega = \int_{\Gamma_2} (p_k - \bar{p}_k) u_k^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_k^* \, d\Gamma \tag{3.179}$$

The next step is to consider that the material is anisotropic but use the isotropic fundamental solution corresponding to a reference elastic model, whose properties may be found by averaging the anisotropic constants. Hence the original coefficients can be expressed as,

$$d_{kjmn} = d^0_{kjmn} + \hat{d}_{kjmn} \tag{3.180}$$

where d^0 indicates the reference state and $\hat{d}$ the residual or difference between the actual and the isotropic elastic constants.

Integrating by parts (3.179) one finds that

$$\begin{aligned} &-\int_{\Omega} \sigma_{kj} \varepsilon^*_{kj} \, d\Omega + \int_{\Omega} b_k u_k^* \, d\Omega \\ &\quad = -\int_{\Gamma_2} \bar{p}_k u_k^* \, d\Gamma - \int_{\Gamma_1} p_k u_k^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_k^* \, d\Gamma \end{aligned} \tag{3.181}$$

Next, one substitutes the following expression

$$\sigma_{kj} = d_{kjmn} \varepsilon_{mn} = (d^0_{kjmn} + \hat{d}_{kjmn}) \varepsilon_{mn} = \sigma^0_{kj} + \hat{\sigma}_{kj} \tag{3.182}$$

while the fundamental solution obeys the following constitutive equation

$$\sigma^*_{kj} = \overset{0}{d}_{kjmn} \varepsilon^*_{mn}$$

Hence formula (3.181) becomes

$$-\int_{\Omega} \sigma^0_{kj}\varepsilon^*_{kj}\,d\Omega - \int_{\Omega} \hat{\sigma}_{jk}\varepsilon^*_{jk}\,d\Omega + \int_{\Omega} b_k u^*_k\,d\Omega$$
$$= -\int_{\Gamma_2} \bar{p}_k u^*_k\,d\Gamma - \int_{\Gamma_1} p_k u^*_k\,d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k)p^*_k\,d\Gamma \tag{3.183}$$

After noticing that

$$\int_{\Omega} \sigma^0_{kj}\varepsilon^*_{kj}\,d\Omega = \int_{\Omega} \sigma^*_{kj}\varepsilon_{kj}\,d\Omega \tag{3.184}$$

one can integrate by parts the first terms of (3.183) to produce

$$\int_{\Omega} \sigma^*_{kj,j}u_k\,d\Omega - \int_{\Omega} \hat{\sigma}_{kj}\varepsilon^*_{kj}\,d\Omega + \int_{\Omega} b_k u^*_k\,d\Omega$$
$$= -\int_{\Gamma_2} \bar{p}_k u^*_k\,d\Gamma - \int_{\Gamma_1} p_k u^*_k\,d\Gamma + \int_{\Gamma_2} u_k p^*_k\,d\Gamma + \int_{\Gamma_1} \bar{u}_k p^*_k\,d\Gamma \tag{3.185}$$

Taking into consideration that the fundamental solution can be applied inside the domain or on the boundary, equation (3.185) gives the following integral statement,

$$c^i_{lk}u^i_k + \int_{\Gamma} p^*_{lk}u_k\,d\Gamma = \int_{\Gamma} u^*_{lk}p_k\,d\Gamma + \int_{\Omega} u^*_{lk}b_k\,d\Omega - \int_{\Omega} \hat{\sigma}_{kj}\varepsilon^*_{lkj}\,d\Omega \tag{3.186}$$

where $\Gamma = \Gamma_1 + \Gamma_2$.

Notice that this formula is similar to (3.79) with only the addition of a new domain term. As before 'l' represents the direction in which the fundamental solution component is acting.

Equation (3.186) contains two types of domain terms. One is due to the body force components b_k and can sometimes be taken to the boundary as described in section 3.6. The other term can be integrated on the domain using cells or transformed into a 'body force' type term and then taken to the boundary. Let us consider the term on its own, i.e.

$$\int_{\Omega} \hat{\sigma}_{kj}\varepsilon^*_{lkj}\,d\Omega \tag{3.187}$$

Next one integrate by parts (3.187) which gives

$$-\int_{\Omega} \hat{\sigma}_{kj,j}u^*_{lk}\,d\Omega + \int_{\Gamma} \hat{p}_k u^*_{lk}\,d\Gamma \tag{3.188}$$

The first integral can now be interpreted in function of a fictitious body force such that

$$\hat{\sigma}_{kj,j} = \hat{b}_k \tag{3.189}$$

Hence equation (3.188) and (3.189) can now be substituted into (3.186) to give

$$c_{ik}^{i}u_k^i + \int_\Gamma p_{ik}^* u_k \, d\Gamma = \int_\Gamma u_{ik}^*(p_k + \hat{p}_k)\, d\Gamma + \int_\Omega u_{ik}^*(b_k + \hat{b}_k)\, d\Omega \tag{3.190}$$

$\hat{p}_k$ and $\hat{b}_k$ are acting now as out of balance forces which represents the anisotropic effect. The problem can be solved iteratively by finding the first solution with $\hat{p}_k$ and $\hat{b}_k \equiv 0$, and then computing their values and resolving the system as many times as required. The influence matrices **H** and **G** are always the same. In matrix form equation (3.190) can be written as,

$$\mathbf{HU} = \mathbf{GP} + \mathbf{G\hat{P}} + \mathbf{DB} + \mathbf{D\hat{B}} \tag{3.191}$$

The equations at the beginning of the iteration process are simply,

$$\mathbf{HU} = \mathbf{GP} + \mathbf{DB} \tag{3.192}$$

and from then on one solves the following equation

$$\mathbf{HU} = \mathbf{GP} + \mathbf{DB} + \mathbf{B}' \tag{3.193}$$

where $\mathbf{B}' = \mathbf{G\hat{P}} + \mathbf{D\hat{B}}$, where $\hat{\mathbf{P}}$, $\hat{\mathbf{B}}$ terms result from the previous step.

While the b_k terms may be reduced to the boundary as shown earlier the main problem remains how to convert $\hat{b}_k$ terms into boundary integrals. If this is not possible one will need to divide the domain into cells as explained in section 3.5.

A general way of reducing body force terms to the boundary is by using particular solutions. This has been generalized in references [16] to [21] by using a technique called the dual reciprocity method (DRM). Although this is an interesting development with many applications in boundary elements it falls beyond the scope of this book and for the further information the reader is referred to the work of Brebbia and his collaborators Nardini, Wrobel and Tang given in the above articles.

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Exercises

3.1. Verify that the equation $\nabla^2(\nabla^2 G_l) + \frac{1}{\mu}\Delta^i e_l = 0$ is obtained by substitution of displacements in terms of Galerkin's vector into Navier's equation for the fundamental solution (equation (3.34)).

3.2. Show that $\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} p_k u_{ik}^* \, d\Gamma = 0$, for any part or complete spherical surface of radius ε around the collocation point.

3.3. Compute the terms of the 3×3 matrix $\mathbf{c}^i$ for a point on a quarter of a sphere edge.

3.4. Obtain the value of the kernel D_{kij} of the internal point stress representation by derivation of the fundamental solution u^*_{lk}.

3.5. Is the matrix $\mathbf{u}^*$ (equation (3.86)) relating two points of the boundless domain symmetric? Answer the same question for matrix $\mathbf{p}^*$ (equation (3.85)). Explain the reasons.

3.6. Assume a rectangular constant element with a centred node for three dimensional problems. Which terms of the matrices $\mathbf{H}^{ii}$ and $\mathbf{G}^{ii}$ relating the node with itself are zero?

3.7. Give an explicit formula, in terms of internal coordinates, for the numerical integration of $\mathbf{u}^*$ and $\mathbf{p}^*$ over a constant rectangular element when the collocation point is not inside the element. The element is defined by the cartesian coordinates of its four corners. 3×3 Gauss quadrature points should be used.

3.8. In section 3.6 it is said that a rigid body motion may be dropped from the fundamental solution when solving two-dimensional elasticity problems. Give the reason why this is possible for bounded regions.

3.9. When will the above question be true for unbounded regions?

3.10. Obtain the 3×3 fundamental solution matrix $\mathbf{u}^*_c = \mathbf{Q}^{iT}\mathbf{u}^*\mathbf{Q}^*$ in cylindrical coordinates when the collocation point is at $\theta' = -\pi/2$.

3.11. When a circular foundation resting on the surface of an elastic half-space is under the effects of a horizontal force, the motion and traction components at any point of the half-space surface may be written as:

$$u_\rho = u'_\rho \cos\theta \qquad t_\rho = t'_\rho \cos\theta$$
$$u_\theta = -u'_\theta \sin\theta \qquad t_\theta = -t'_\theta \sin\theta$$
$$u_z = u'_z \cos\theta \qquad t_z = t'_z \cos\theta$$

Write the integral equation relating u'_ρ, u'_θ, u'_z and t'_ρ, t'_θ, t'_z for boundary points using equation (3.170) and the formula derived in exercise 3.10, for the θ component.