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Tutorial 5:
FUNDAMENTAL SOLUTIONS:
II-MATRIX OPERATORS

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Summary and objectives

In this tutorial, we will continue the discussion, started in the tutorial 4, about the derivation of the fundamental solutions. In the former tutorial, we presented techniques for setting up the fundamental solution for simple and compound operators. Herein, we will discuss the use of operator decoupling technique to breakdown matrix operators to simple or compound scalar ones. This method is due to Lars Hörmander [1] and it was introduced to the setting up of fundamental solutions within the context of boundary element method by Kitahara (cf. Ref. [2]) and Tosaka (cf. Ref. [3]).

We will present two examples: elasticity and plate bending problems, to demonstrate the usage of such technique.

1 The cofactor matrix

The understanding of Hörmander method is easy as it is mainly dependant on understanding of simple definitions in matrix algebra. In this section we will review some of these basic algebra. Consider the following matrix:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1}$$

The cofactor of any element is defined a s follows:

$$^{co}a_{ij} = (-1)^{(i+j)} \times \left| \text{minor of } a_{ij} \right| \tag{2}$$

for example:

$$^{co}a_{11} = (+)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \tag{3}$$

Similarly, one can compute:

$$^{co}a_{12} = (-)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \tag{4}$$

and so on …

The overall matrix, which composed of the cofactor elements, is called the cofactor matrix. The same procedures can be done if the matrix [a] has a dimension higher or smaller than 3×3.

2 Hörmander method

Hörmander method is a method generally used not to derive the fundamental solution but mainly to decompose matrix operators to simple scalar operators, which their fundamental solutions can be obtained easily (recall Primer 4). Once the fundamental solutions or the scalar potentials for these simple operators are obtained, Hörmander technique provides a backward procedure to construct the fundamental solution for the original matrix operator. In order to demonstrate the general procedures of Hörmander method, consider the following general governing differential equation of a certain problem:

$$Lu = b \tag{5}$$

where L is matrix-type differential operator, b is the body force vector and u the problem density. It is required to set up the fundamental solution U of this problem in order to be used within the relevant boundary element formulation. The steps of Hörmander method are as follows:

1- Compute the adjoint operator: Recall tutorial 2, after setting up the boundary integral formulation, we need to compute the fundamental solution of the following equation:

$$L^{adj}U = -\delta \tag{6}$$

where L^{adj} is the adjoint operator of the original operator L, δ is the Dirac delta and U is the desirable fundamental solution. If the operator L^{adj} is a simple or compound operator, we can use techniques described in tutorial 4 to obtain the fundamental solution. Herein we are going to consider L^{adj} as a matrix operator.

2- Compute the cofactor matrix of the adjoint operator $^{co}L^{adj}$ using the way described in the former section for general matrix.

3- Compute the transpose of the cofactor matrix $\left({}^{co}L^{adj}\right)^T$

4- Compute the determinant of the transpose of the cofactor matrix (note that the determinant always is an scalar operator): $\det\left|\left({}^{co}L^{adj}\right)^T\right|$. It has to be noted that the determinant can be computed using any known techniques from traditional algebra. However, in many cases, this could be achieved easier by computing the scalar multiplication (dot product) of two corresponding rows or columns in the matrices of: $\left({}^{co}L^{adj}\right)^T$ and the matrix of the original operator L.

5- Compute the scalar potential Φ which is the solution of the following equation:

$$\det\left|\left({}^{co}L^{adj}\right)^T\right|\Phi = -\delta \tag{7}$$

It has to be noted that, up to this step, it can be seen that instead of computing the fundamental solution for the original operator L, which is difficult, we decomposed this operator to a new scalar operator: $\det\left|\left({}^{co}L^{adj}\right)^T\right|$ which is a simple or compound operator, and can be dealt with using the methods presented in tutorial 4. In the next step we will construct the fundamental solution U using the obtained scalar potential Φ.

6- Compute the fundamental solution using the following equation.

$$U = {}^{co}L^{adj}\Phi \tag{8}$$

It has to be noted that, the Hörmander method, generally, represents the fundamental solution U as vector derivatives of another scalar potential Φ. This could be used in transforming domain integrals to the boundary as will be discussed in later tutorials.

3 Example to elasticity

In this section we will be presenting the derivation of the elasticity fundamental solution using the Hörmander technique. Consider the following Navier governing differential equations [4]:

$$L_{ij}U^*_{kj} = -\delta(\boldsymbol{\xi}, \mathbf{x})\delta_{ki} \tag{9}$$

where

$$L_{ij} = G\nabla^2\delta_{ij} + \frac{G}{1-2\nu}\partial_i\partial_j \tag{10}$$

or

$$L = \begin{bmatrix} G\nabla^2 + \frac{G}{1-2\nu}\partial_1\partial_1 & \frac{G}{1-2\nu}\partial_1\partial_2 \\ \frac{G}{1-2\nu}\partial_2\partial_1 & G\nabla^2 + \frac{G}{1-2\nu}\partial_2\partial_2 \end{bmatrix} \tag{11}$$

which represents the original differential operator of the problem. G denotes the modulus of rigidity and ν is the Poisson's ratio. This operator is a self-adjoint operator; therefore:

$$L_{ij}^{adj} = L_{ij} \tag{12}$$

Following Hörmander, the cofactor matrix of the adjoint operator can be obtained as follows [5]:

$$^{co}L^{adj} = \begin{bmatrix} G\nabla^2 + \frac{G}{1-2\nu}\partial_2\partial_2 & -\frac{G}{1-2\nu}\partial_2\partial_1 \\ -\frac{G}{1-2\nu}\partial_1\partial_2 & G\nabla^2 + \frac{G}{1-2\nu}\partial_1\partial_1 \end{bmatrix} \tag{13}$$

which can be written, in an indicial notation as follows:

$$^{co}L_{ij}^{adj} = \frac{G}{1-2\nu}\left[2(1-\nu)\delta_{ij}\nabla^2 - \partial_i\partial_j\right] \tag{14}$$

The transpose of the cofactor matrix can be written as follows:

$$\left(^{co}L_{ij}^{adj}\right)^T = {}^{co}L_{ij}^{adj} \tag{15}$$

and its relevant determinant can be computed as follows:

$$\det\left|\left(^{co}L_{ij}^{adj}\right)^T\right| = \left(G\nabla^2 + \frac{G}{1-2\nu}\partial_2\partial_2\right)\left(G\nabla^2 + \frac{G}{1-2\nu}\partial_1\partial_1\right) - \left(\frac{-G}{1-2\nu}\partial_2\partial_1\right)\left(\frac{-G}{1-2\nu}\partial_1\partial_2\right) \tag{16}$$

noting that:

$$\partial_i\partial_j = \partial_j\partial_i \tag{17}$$

and

$$\nabla^2 = \partial_1\partial_1 + \partial_2\partial_2 \tag{18}$$

Then one can equation (16) as:

$$\det\left|\left(^{co}L_{ij}^{adj}\right)^T\right| = \frac{2G^2(1-\nu)}{1-2\nu}\nabla^4 \tag{19}$$

According to Hörmander, we need to obtain a suitable scalar potential Φ that satisfies the following equation:

$$\frac{2G^2(1-\nu)}{1-2\nu}\nabla^4\Phi(\boldsymbol{\xi},\mathbf{x}) = -\delta(\boldsymbol{\xi},\mathbf{x}) \tag{20}$$

A suitable singular particular solution can be obtained as follows (Recall tutorial 4):

$$\Phi = \frac{-(1-2\nu)}{2G^2(1-\nu)}\frac{1}{8\pi}r^2 \ln r + f \tag{21}$$

where:

$$f = a\,r^2 + b\ln r + c \tag{22}$$

in which, a,b,c are arbitrary constants. It has to be noted that Φ, herein, represents the well-known Galerkin tensor [4] and f is just a complementary solution for the bi-harmonic operator, which can be omitted by setting all constants to be zeros. From the last step of Hörmander method, one can write the fundamental solution as follows (see equation (8)):

$$U_{ij} = \frac{G}{1-2\nu}\left[2(1-\nu)\delta_{ij}\Phi,_{\theta\theta} - \Phi,_{ij}\right] \tag{23}$$

The corresponding derivatives of Φ and f are:

$$f,_i = r,_i\left(2\,a\,r + \frac{b}{r}\right) \tag{24}$$

$$f,_{ij} = \frac{-2br,_i\, r,_j}{r^2} + \left(2\,a\,r + \frac{b}{r}\right)\delta_{ij} \tag{25}$$

$$\nabla^2 f = 4a \tag{26}$$

Substituting in equation (23), one can get:

$$U_{ij} = U_{ij}^{(1)} + U_{ij}^{(2)} \tag{27}$$

where:

$$U_{ij}^{(1)} = \frac{1}{8\pi G(1-\nu)}\left\{-\delta_{ij}\left[\frac{7-8\nu}{2} + (3-4\nu)\ln r\right] + r,_i\, r,_j\right\} \tag{28}$$

and

$$U_{ij}^{(2)} = \frac{G}{1-2\nu}\left\{2a(3-4\nu)\delta_{ij} - \frac{b}{r^2}\left[\delta_{ij} + 2r,_i\, r,_j\right]\right\} \tag{29}$$

If a and b are chosen to be:

$$a = \frac{(1-2\nu)(7-8\nu)}{32\pi G^2 (1-\nu)(3-4\nu)} \qquad \text{and} \qquad b = 0 \tag{30}$$

the well-know Kelvin fundamental solution expressions can be obtained as follows [4]:

$$U_{ij} = \frac{1}{8\pi G(1-\nu)} \left\{ -\delta_{ij}(3-4\nu)\ln r + r,_i \, r,_j \right\} \tag{31}$$

4 Example to plates in bending

1- The original operator of the problem is given as [6]:

$$L = \begin{bmatrix} \frac{D(1-\nu)}{2}\left[\left(\nabla^2 - \lambda^2\right) + \frac{1+\nu}{1-\nu}\partial_1\partial_1\right] & \frac{D(1+\nu)}{2}\partial_1\partial_2 & -\frac{D(1-\nu)}{2}\lambda^2\partial_1 \\ \frac{D(1+\nu)}{2}\partial_1\partial_2 & \frac{D(1-\nu)}{2}\left[\left(\nabla^2 - \lambda^2\right) + \frac{1+\nu}{1-\nu}\partial_2\partial_2\right] & -\frac{D(1-\nu)}{2}\lambda^2\partial_2 \\ \frac{D(1-\nu)}{2}\lambda^2\partial_1 & \frac{D(1-\nu)}{2}\lambda^2\partial_2 & \frac{D(1-\nu)}{2}\lambda^2\nabla^2 \end{bmatrix} \tag{32}$$

where D is the plate modulus and λ is the shear factor.

2- The adjoint operator can be obtained as follows. Noting that this operator is not a self-adjoint operator:

$$L^{adj} = \begin{bmatrix} \frac{D(1-\nu)}{2}\left[\left(\nabla^2 - \lambda^2\right) + \frac{1+\nu}{1-\nu}\partial_1\partial_1\right] & \frac{D(1+\nu)}{2}\partial_1\partial_2 & \frac{D(1-\nu)}{2}\lambda^2\partial_1 \\ \frac{D(1+\nu)}{2}\partial_1\partial_2 & \frac{D(1-\nu)}{2}\left[\left(\nabla^2 - \lambda^2\right) + \frac{1+\nu}{1-\nu}\partial_2\partial_2\right] & \frac{D(1-\nu)}{2}\lambda^2\partial_2 \\ -\frac{D(1-\nu)}{2}\lambda^2\partial_1 & -\frac{D(1-\nu)}{2}\lambda^2\partial_2 & \frac{D(1-\nu)}{2}\lambda^2\nabla^2 \end{bmatrix} \tag{33}$$

3- The elements of the cofactor matrix of the L^{adj} can be computed as follows [7]:

$$\begin{aligned} {}^{co}L_{11}^{adj} &= (+)\left\{ \frac{D^2(1-\nu)^2}{4}\lambda^2\nabla^2\left[\left(\nabla^2 - \lambda^2\right) + \frac{1+\nu}{1-\nu}\partial_2\partial_2\right] + \frac{D^2(1-\nu)^2}{4}\lambda^4\partial_2\partial_2 \right\} \\ &= \frac{D^2(1-\nu)^2}{4}\lambda^2\left[\nabla^4 - \lambda^2\partial_1\partial_1 + \frac{1+\nu}{1-\nu}\partial_2\partial_2\nabla^2\right] \end{aligned} \tag{34}$$

$$ {}^{co}L_{22}^{adj} = (+)\left\{\frac{D^2(1-\nu)^2}{4}\lambda^2\nabla^2\left[\left(\nabla^2-\lambda^2\right)+\frac{1+\nu}{1-\nu}\partial_1\partial_1\right]+\frac{D^2(1-\nu)^2}{4}\lambda^4\partial_1\partial_1\right\} $$

$$ = \frac{D^2(1-\nu)^2}{4}\lambda^2\left[\nabla^4-\lambda^2\partial_2\partial_2+\frac{1+\nu}{1-\nu}\partial_1\partial_1\nabla^2\right] \qquad (35) $$

$$ {}^{co}L_{33}^{adj} = (+)\left\{\frac{D^2(1-\nu)^2}{4}\left[\left(\nabla^2-\lambda^2\right)+\frac{1+\nu}{1-\nu}\partial_1\partial_1\right]\left[\left(\nabla^2-\lambda^2\right)+\frac{1+\nu}{1-\nu}\partial_1\partial_1\right]-\frac{D^2(1+\nu)^2}{4}\partial_1\partial_2\partial_1\partial_2\right\} $$

$$ = \frac{D^2(1-\nu)}{4}\left[2\nabla^4+(\nu-3)\lambda^2\nabla^2+(1-\nu)\lambda^4\right] \qquad (36) $$

$$ {}^{co}L_{12}^{adj} = (-)\left\{\frac{D(1+\nu)}{2}\partial_1\partial_2\,\frac{D(1-\nu)}{2}\lambda^2\nabla^2+\frac{D(1-\nu)}{2}\lambda^2\partial_2\,\frac{D(1-\nu)}{2}\lambda^2\partial_1\right\} $$

$$ = \frac{D^2(1-\nu)}{4}\lambda^2\partial_1\partial_2\left[(1+\nu)\nabla^2+(1-\nu)\lambda^2\right] \qquad (37) $$

$$ {}^{co}L_{21}^{adj} = -\frac{D^2(1-\nu)}{4}\lambda^2\partial_1\partial_2\left[(1+\nu)\nabla^2+(1-\nu)\lambda^2\right] \qquad (38) $$

$$ {}^{co}L_{31}^{adj} = (+)\left\{\frac{D(1+\nu)}{2}\partial_1\partial_2\,\frac{D(1-\nu)}{2}\lambda^2\partial_2-\frac{D(1-\nu)}{2}\lambda^2\partial_1\,\frac{D(1-\nu)}{2}\left[\left(\nabla^2-\lambda^2\right)+\frac{1+\nu}{1-\nu}\partial_2\partial_2\right]\right\} $$

$$ = \frac{D^2(1-\nu)^2}{4}\lambda^2\partial_1\left(\nabla^2-\lambda^2\right) \qquad (39) $$

$$ {}^{co}L_{13}^{adj} = -\frac{D^2(1-\nu)^2}{4}\lambda^2\partial_1\left(\nabla^2-\lambda^2\right) \qquad (40) $$

$$ {}^{co}L_{23}^{adj} = (-)\left\{\frac{D^2(1-\nu)^2}{4}\left[\left(\nabla^2-\lambda^2\right)+\frac{1+\nu}{1-\nu}\partial_1\partial_1\right]\lambda^2\partial_2-\frac{D^2(1-\nu^2)}{4}\lambda^2\partial_1\partial_1\partial_2\right\} $$

$$ = -\frac{D^2(1-\nu)^2}{4}\lambda^2\partial_2\left(\nabla^2-\lambda^2\right) \qquad (41) $$

$$
{}^{co}L_{32}^{adj} = \frac{D^2(1-\nu)^2}{4}\lambda^2\partial_2\left(\nabla^2 - \lambda^2\right) \tag{42}
$$

Then the ${}^{co}L^{adj}$ can be written as:

$$
{}^{co}L^{adj} = \begin{bmatrix} {}^{co}L_{\alpha\beta}^{adj} & {}^{co}L_{\alpha3}^{adj} \\ {}^{co}L_{3\beta}^{adj} & {}^{co}L_{33}^{adj} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} {}^{co}L_{11}^{adj} & {}^{co}L_{12}^{adj} \\ {}^{co}L_{21}^{adj} & {}^{co}L_{22}^{adj} \end{bmatrix} & \begin{bmatrix} {}^{co}L_{13}^{adj} \\ {}^{co}L_{32}^{adj} \end{bmatrix} \\ \begin{bmatrix} {}^{co}L_{31}^{adj} & {}^{co}L_{32}^{adj} \end{bmatrix} & {}^{co}L_{33}^{adj} \end{bmatrix} \tag{43}
$$

which can be re-written in indicial notation as follows [7]:

$$
{}^{co}L_{\alpha\beta}^{adj} = \frac{D^2(1-\nu)^2}{4}\lambda^2\left[\nabla^4\delta_{\alpha\beta} - \lambda^2\partial_\alpha\partial_\beta + \frac{1+\nu}{1-\nu}\nabla^2\left(\nabla^2\delta_{\alpha\beta} - \partial_\alpha\partial_\beta\right)\right] \tag{44}
$$

$$
{}^{co}L_{\alpha3}^{adj} = \frac{D^2(1-\nu)^2}{4}\lambda^2\partial_\alpha\left(\nabla^2 - \lambda^2\right) \tag{45}
$$

$$
{}^{co}L_{3\beta}^{adj} = -\frac{D^2(1-\nu)^2}{4}\lambda^2\partial_\beta\left(\nabla^2 - \lambda^2\right) = -{}^{co}L_{\alpha3}^{adj}\delta_{\alpha\beta} \tag{46}
$$

$$
{}^{co}L_{33}^{adj} = \frac{D^2(1-\nu)}{4}\left[2\nabla^4 + (\nu-3)\lambda^2\nabla^2 + (1-\nu)\lambda^4\right] \tag{47}
$$

4- The determinant of the transpose of the cofactor matrix can be computed using the multiplication of two corresponding rows or columns in the matrices. For example, if the multiplications of the two second rows is considered, it gives:

$$
\begin{aligned}
\det\left|\left({}^{co}L^{adj}\right)^T\right| &= \left[\frac{D^2(1-\nu)^2}{4}\lambda^2\partial_\beta\left(\nabla^2 - \lambda^2\right)\right] \times \left[\frac{D\ (1-\nu)}{2}\lambda^2\partial_\beta\right] + \\
&\quad \left[\frac{D^2(1-\nu)}{4}\left[2\nabla^4 + (\nu-3)\lambda^2\nabla^2 + (1-\nu)\lambda^4\right]\right] \times \left[\frac{D\ (1-\nu)}{2}\lambda^2\nabla^2\right] \\
&= \frac{D^3(1-\nu)^2}{4}\lambda^2\nabla^4\left(\nabla^2 - \lambda^2\right)
\end{aligned} \tag{48}
$$

It has to be noted that the same results can be obtained if the two first rows or the two first columns are considered instead. Recalling equation (7), we seek the fundamental solution of the scalar potential Φ, which is the solution of the following equation:

$$\frac{D^3(1-\nu)^2}{4}\lambda^2\nabla^4\left(\nabla^2-\lambda^2\right)\Phi(\xi,\mathbf{x})=-\delta(\xi,\mathbf{x}) \tag{49}$$

A suitable singular particular solution can be obtained using the partial fraction analogy (Recall tutorial 4) as follows [6]:

$$\begin{aligned}\Phi(\xi,\mathbf{x})&=\frac{-4}{D^3(1-\nu)^2\lambda^2}\left\{\frac{-1}{\lambda^2\nabla^4}-\frac{1}{\lambda^4\nabla^2}+\frac{1}{\lambda^4\left(\nabla^2-\lambda^2\right)}\right\}\delta(\xi,\mathbf{x})\\&=\frac{-4}{D^3(1-\nu)^2\lambda^2}\left\{\frac{-1}{\lambda^2}\left(\frac{-1}{8\pi}r^2\ln r\right)-\frac{1}{\lambda^4}\left(\frac{-1}{2\pi}\ln r\right)+\frac{1}{\lambda^4}\left(\frac{-1}{2\pi}K_0(\lambda r)\right)\right\}\end{aligned} \tag{50}$$

in which K_0 is a modified Bessel function.

5- The final fundamental solution of the problem can be obtained using equation (8) as follows:

$$U_{ij}={}^{co}L_{ij}^{adj}\Phi \tag{51}$$

or

$$U_{\alpha\beta}=\frac{1}{8\pi D(1-\nu)}\left\{\left[8B(\lambda r)-(1-\nu)\left(2\ln(\lambda r)-1\right)\right]\delta_{\alpha\beta}-\left[8A(\lambda r)+2(1-\nu)\right]r,_{\alpha}r,_{\beta}\right\} \tag{52}$$

$$U_{\alpha 3}=-U_{3\alpha}=\frac{1}{8\pi D}\left(2\ln(\lambda r)-1\right)r\,r,_{\alpha} \tag{53}$$

$$U_{33}=\frac{1}{8\pi D(1-\nu)\lambda^2}\left[(1-\nu)\lambda^2r^2\left(2\ln(\lambda r)-1\right)-8\ln(\lambda r)\right] \tag{54}$$

where

$$A(\lambda r)=K_0(\lambda r)+\frac{2}{\lambda r}\left[K_1(\lambda r)-\frac{1}{\lambda r}\right] \tag{55}$$

$$B(\lambda r)=K_0(\lambda r)+\frac{1}{\lambda r}\left[K_1(\lambda r)-\frac{1}{\lambda r}\right] \tag{56}$$

5 Conclusions

In this tutorial, we have presented how to use the Hörmander technique in decomposing matrix operators to simple scalar operators. Two examples were presented, one for the elasticity problem and the other for plates in bending. In the next tutorials we will discuss in details the relevant algebra that used to obtain the final form of the fundamental solutions. Also the derivation of the corresponding traction kernels will be also discussed.

6 Solution of the exercise in tutorial 4

6.1 Exercise 1

The solution of this exercise is give in Ref. [8] page 370.

6.2 Exercise 2

The fundamental solution U of the following equation:

$$\nabla^4(\nabla^2 - a^2)U = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{57}$$

can be easily obtained using partial fraction analogy as follows:

$$\begin{aligned} U(\boldsymbol{\xi}, \mathbf{x}) &= -\left\{ \frac{-1}{a^2\nabla^4} - \frac{1}{a^4\nabla^2} + \frac{1}{a^4\left(\nabla^2 - a^2\right)} \right\} \delta(\boldsymbol{\xi}, \mathbf{x}) \\ &= \frac{1}{a^2}\left(\frac{-1}{8\pi} r^2 \ln r\right) + \frac{1}{a^4}\left(\frac{-1}{2\pi} \ln r\right) - \frac{1}{a^4}\left(\frac{-1}{2\pi} K_0(ar)\right) \end{aligned} \tag{58}$$

which is similar to what we did in equation (50).

References and Further Reading

[1] Hörmander, L., Linear Partial Differential Operators, Springer-Verlag, Berlin, 1963.

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[3] Tosaka, N., Integral equation formulations for viscous flow problems, Proceedings of the 2nd Japan National Symposium on Boundary Element Methods, Tokyo, 1985.

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