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Tutorial 4:
FUNDAMENTAL SOLUTIONS: I-SIMPLE AND COMPOUND OPERATORS

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Summary and objectives

In the tutorial 3, we presented other examples on the derivation of the boundary integral equation in the direct form. Mainly, elasticity and plate in bending problems were discussed. In this tutorial, we will discuss the definitions and the methods of derivation of fundamental solutions. The use of such solution within the boundary element method was discussed in the former tutorial. A table presents the commonly used forms of fundamental solution is given. Also a method based on simple analogy to the algebraic partial fraction is discussed to decompose compound differential operators. In the next tutorial, we will continue discussing how to set up the fundamental solutions for complex matrix operators.

1 Definitions

The fundamental solution can be defined in the most simple way as the response due to unit source in an infinite problem. For example when we say $U_{ij}(\xi,\mathbf{x})$ is the fundamental solution for displacements in elasticity problems that means: $U_{ij}(\xi,\mathbf{x})$ is the displacement at point $\mathbf{x}$ in the direction j due to unit point load applied at ξ in the i direction. It can be seen that $U_{ij}(\xi,\mathbf{x})$ is a kernel between two-points. From Betti reciprocal theory it is clear that $U_{ij}(\xi,\mathbf{x})=U_{ji}(\xi,\mathbf{x})$.

Mathematically the fundamental solution of a problem is the solution of the governing differential equation when the Dirac delta is acting as a forcing term, appears of the right hand side [1]. It has to be noted that no boundary conditions is forced to simulate the infinite nature of the problem. In other words, it is the particular solution of the problem corresponding to the Dirac delta distribution. Provided that the Dirac delta posses the singular nature, the fundamental solution is also singular. The name "fundamental" came from the fact that it is the solution of the most "fundamental" problem in mechanics, which deals with a unit source in an infinite body. It is called also the principal solution; the singular solution or the free-space Green's function. From this definitions, the fundamental solution can be defined as follows:

$$LU(\xi,\mathbf{x}) = -\delta(\xi,\mathbf{x}) \tag{1}$$

where L is a scalar differential operator and $\delta(\xi,\mathbf{x})$ is the Paul Dirac delta, in which is the ξ source point and the $\mathbf{x}$ is a field point. If L is a matrix operator, equation (1) can be re-written as follows:

$$L_{ij}U^{*}_{kj}(\boldsymbol{\xi},\mathbf{x}) = -\delta(\boldsymbol{\xi},\mathbf{x})\delta_{ki} \tag{2}$$

It has to be noted that no boundary conditions are enforced in both equations (1) and (2). In this tutorial we will discuss the derivation of the fundamental solution in equation (1) whereas the derivation of fundamental solutions for matrix operators as that of equation (2) will be discussed in the next tutorial.

2 Useful properties of the Paul Dirac delta

The following properties of the Paul Dirac delta could be used in the derivation of the fundamental solution [1]:

$$\delta(\boldsymbol{\xi},\mathbf{x}) = \begin{cases} \infty & \text{when } \boldsymbol{\xi} = \mathbf{x} \\ 0 & \text{when } \boldsymbol{\xi} \neq \mathbf{x} \end{cases} \tag{3}$$

$$\lim_{\Omega\to\mathbf{0}} \int_{\Omega} \delta(\xi,\mathbf{x})\,d\Omega = 1 \tag{4}$$

$$\int_{\Omega} \delta(\xi,\mathbf{x})F(\mathbf{x})\,d\Omega = F(\xi) \tag{5}$$

in which Ω is an arbitrary domain. For simplicity, it will be chosen as either a circle or sphere for two- and three-dimensional problems, respectively [2]. It has to be noted that we already have made use of the third property (equation (5)) in the former two tutorials. In this tutorial the second property (equation (4)) will help in the steps of the fundamental solution derivation.

3 Methods of derivation

Derivation of fundamental solutions is a lengthy task for difficult operators. However, in many cases, this could be a systematic procedure. The general technique for deriving the fundamental solution is to use integral transforms, such as, Fourier, Laplace or Hankel transforms [1]. Such a technique involves complicated mathematics and has very sophisticated procedures. Therefore it will be covered in latter tutorial as an advanced topic. In the next section, we will demonstrate the derivation of the fundamental solution for well-known simple operators such as the Laplacian. The procedures will be described for both two- and three-dimensional problems. In section 5, a table will be presented to summarize the fundamental solutions for commonly used simple operators. Then in section 6, a simple analogy to the algebraic partial fraction technique will be used to decompose compound operators to the simple forms presented in section 5.

4 Fundamental solutions using integration

In this section we will demonstrate a technique based on direct integration and making use of the properties of the Dirac delta to construct the fundamental solution. The starting point of this technique is to solve the following homogeneous equation:

$$\mathrm{L}U(\boldsymbol{\xi}, \mathbf{x}) = 0 \qquad (6)$$

This equation can be solved using any simple technique in the calculus such as, direct integration in polar coordinate, separation of variables, variation of parameters or using complex variable transformation (for the case of two dimensional problems only). Some constants will appear due to the integration procedures. In order to obtain such constants, we can make use of the Dirac Delta property in equation (4) after combining it with equation (1), to give:

$$\lim_{\Omega \to \mathbf{0}} \int_{\Omega} \mathrm{L}U(\xi, \mathbf{x}) d\Omega = -1 \qquad (7)$$

As the domain Ω can be chosen arbitrary, one can present it as a small circle (sphere for the three-dimensional case) of a radius ε, for simplicity. Then the limit in equation (7) will be performed as $\varepsilon \rightarrow 0$. It has to be noted that the integration in equation (7) could be performed easily by transforming it to the boundary of the circle using the integration by parts procedures presented in tutorial 2. More details about this method is presented by Rahman in Ref. [2]. The following examples will demonstrate that idea.

4.1 Laplace operator in two-dimension

Consider the Laplace equation in two-dimension:

$$\nabla^2 U(\xi, \mathbf{x}) = -\delta(\xi, \mathbf{x}) \qquad (8)$$

The first step to construct the fundamental solution is to solve the following homogeneous equation:

$$\nabla^2 U(\xi, \mathbf{x}) = 0 \qquad (9)$$

Or in polar coordinate it could be expressed as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) U(\xi, \mathbf{x}) = 0 \qquad (10)$$

By integration, one can obtain:

$$U(\xi, \mathbf{x}) = a + b \ln r \qquad (11)$$

In order to obtain the constants a and b in equation (11) we will make use of equation (7), to give:

$$\lim_{\Omega \to 0} \int_{\Omega} \nabla^2 U(\xi, \mathbf{x}) d\Omega = -1 \qquad (12)$$

or

$$\lim_{\Omega \to 0} \int_{\Omega} \nabla^2 \left(a + b \ln r\right) d\Omega = -1 \qquad (13)$$

Applying the integration by parts (or Green's second identity), it gives:

$$\lim_{\varepsilon \to 0} \int_{\Gamma} \frac{\partial}{\partial n} \left(a + b \ln r\right) d\Gamma = -1 \qquad (14)$$

Use the polar coordinate notation, where $d\Gamma = r d\theta$, one can obtain:

$$\lim_{\varepsilon \to 0} \int_{\theta=0}^{\theta=2\pi} \frac{\partial}{\partial n} \left(a + b \ln \varepsilon\right) d\Gamma = -1 \qquad (15)$$

Noting that for a circle $\frac{\partial \varepsilon}{\partial n} = 1$, then we can obtain: a is an arbitrary constant and $b = \frac{1}{2\pi}$. The final form of the fundamental solution can be written as:

$$U(\xi, \mathbf{x}) = a + \frac{1}{2\pi} \ln r \qquad (16)$$

It has to be noted that, in carrying out the integration in the former example, the following alternative transformation could be used [3]:

$$z = x_1 + i\, x_2 \quad \text{and} \quad \bar{z} = x_1 - i\, x_2 \qquad (17)$$

and it is easy to prove that:

$$x_1 = \frac{z + \bar{z}}{2} \quad \text{and} \quad x_2 = \frac{z - \bar{z}}{2\, i} \qquad (18)$$

4.2 Laplace operator in the three-dimension

Similar to the two-dimensional case, Equation (9) can be re-written as:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) U(\xi, \mathbf{x}) = 0 \qquad (19)$$

By integration one can obtain:

$$U(\xi, \mathbf{x}) = a + \frac{b}{r} \tag{20}$$

By satisfying the condition in equation (7), one can obtain:

$$\lim_{\varepsilon \to 0} \int_{\gamma=0}^{\gamma=2\pi} \int_{\theta=0}^{\theta=2\pi} \frac{\partial}{\partial n}\left(a + \frac{b}{\varepsilon} \right) \varepsilon^2 d\theta d\gamma = -1 \tag{21}$$

Noting that for a sphere $\frac{\partial \varepsilon}{\partial n} = 1$ and $d\Gamma = r^2 d\theta d\gamma$, then we can easy obtain: a is arbitrary constant and $b = \frac{1}{4\pi}$. Then the final form of the fundamental solution can be written as:

$$U(\xi, \mathbf{x}) = a + \frac{1}{4\pi r} \tag{22}$$

5 Common fundamental solution

The most commonly fundamental solutions are used as basics for many problems in computational mechanics are presented in Table 1 for one-, two- and three-dimensional case [4].

Table 1: Fundamental solutions for most commonly used operators.

Equation	One-dimensional	Two-dimensional	Three-dimensional
Laplace $\nabla^2 U = -\delta(\xi, \mathbf{x})$	$U = -\frac{\lvert x \rvert}{2}$	$U = -\frac{1}{2\pi} \ln r$	$U = -\frac{1}{4\pi r}$
Helmholtz $\left(\nabla^2 + \lambda^2\right) U = -\delta(\xi, \mathbf{x})$	$U = -\frac{1}{2\lambda} \sin(\lambda \lvert x \rvert)$	$U = -\frac{1}{4i} H^{(1)}(\lambda r)$	$U = -\frac{1}{4\pi r} e^{-i\lambda r}$
Modified Helmholtz $\left(\nabla^2 - \lambda^2\right) U = -\delta(\xi, \mathbf{x})$	$U = -\frac{1}{2\lambda} \sin(-i\lambda \lvert x \rvert)$	$U = -\frac{1}{2\pi} K_0(\lambda r)$	$U = -\frac{1}{4\pi r} e^{\lambda r}$
Bi-harmonic $\nabla^4 U = -\delta(\xi, \mathbf{x})$		$U = -\frac{1}{8\pi} r^2 \ln r$	

Where $H^{(1)}$ and K_0 are Hankel and Bessel functions respectively.

6 Partial fraction analogy

In the former section, we have demonstrated simple procedures to derive the fundamental solution for simple operators such as the Laplacian. In this section we will present a technique based on an analogy to the algebraic partial fraction to decompose compound operator to simple operators of the forms given in Table 1. In order to demonstrate this method, we will consider the following examples:

6.1 Example 1:

Consider if we want to construct the fundamental solution of the following compound operator:

$$(\nabla^2 - a^2)(\nabla^2 - b^2)U = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{23}$$

From Table 1, we can obtain the following fundamental solutions U_1 and U_2, where:

$$(\nabla^2 - a^2)U_1 = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{24}$$

and

$$(\nabla^2 - b^2)U_2 = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{25}$$

In order to obtain the fundamental solution U in terms of U1 and U2, we try to make an analogy to the partial fraction theory, as follows:

$$U = \frac{-\delta(\boldsymbol{\xi}, \mathbf{x})}{(\nabla^2 - a^2)(\nabla^2 - b^2)} \tag{26}$$

$$= \frac{-\delta(\boldsymbol{\xi}, \mathbf{x})}{(a^2 - b^2)(\nabla^2 - a^2)} + \frac{-\delta(\boldsymbol{\xi}, \mathbf{x})}{(b^2 - a^2)(\nabla^2 - b^2)} \tag{27}$$

$$= \frac{U_1}{(a^2 - b^2)} + \frac{U_2}{(b^2 - a^2)} \tag{28}$$

6.2 Example 2:

The following example will be considered to make the idea more clear. Consider that we want to construct the fundamental solution of the following operator:

$$\nabla^2(\nabla^2 - a^2)U = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{29}$$

From Table 1, we can obtain fundamental solutions U_1 and U_2 where:

$$\nabla^2 U_1 = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{30}$$

and

$$(\nabla^2 - a^2)U_2 = -\delta(\boldsymbol{\xi}, \mathbf{x}) \tag{31}$$

Following the same analogy of the previous example, one can write:

$$U = \frac{-\delta(\xi, \mathbf{x})}{\nabla^2(\nabla^2 - a^2)} \tag{32}$$

$$= \frac{-\delta(\xi, \mathbf{x})}{-a^2\nabla^2} + \frac{-\delta(\xi, \mathbf{x})}{a^2(\nabla^2 - a^2)} \tag{33}$$

$$= \frac{U_1}{-a^2} + \frac{U_2}{a^2} \tag{34}$$

As it can be seen that, the partial fraction analogy helps in decomposing compound scalar operators to well-known simple operators.

7 Conclusions

In this tutorial, we have demonstrated simple procedures for deriving the fundamental solutions for well know and commonly used differential operators in computational mechanics. We also covered a technique based on an analogy to the simple concept of algebraic partial fractions to obtain the fundamental solution for compound operators consist of product of the well-know simple operators. In the next tutorial we will cover the use of Hörmander method to decouple complex operators. We will give many examples to show the step-by-step derivation of the fundamental solution kernels.

8 Exercise:

1- Using the method described in section 4, obtain the fundamental solution for the modified Helmholtz equation.

2- Use the partial fraction analogy, find the fundamental solution for the following operator:

$$\nabla^4\left(\nabla^2 - a^2\right)U = -\delta(\xi, \mathbf{x}) \tag{35}$$

9 Solution of the exercise in tutorial 3

In order to derive the direct boundary integral equation for the shear-deformable plate bending resting on the two-parameter Pasternak foundation model, the following identity can be written:

$$\int_\Omega \left[\left(M_{\alpha\beta},_\beta - Q_\alpha\right)U_\alpha + \left(Q_\alpha,_\alpha + q - K_f u_3 + G_f \nabla^2 u_3\right)U_3\right] d\Omega = 0 \tag{36}$$

or

$$\int_\Omega \left[\left(M_{\alpha\beta},_\beta - Q_\alpha\right)U_\alpha + \left(Q_\alpha,_\alpha + q\right)U_3\right] d\Omega + \int_\Omega \left(-K_f u_3 + G_f \nabla^2 u_3\right)U_3 \, d\Omega = 0 \tag{37}$$

The first integral will lead to the same results as described in Tutorial 3. Now, we will consider the second integral:

$$\int_{\Omega}\left(-K_f u_3 + G_f \nabla^2 u_3\right)U_3\, d\Omega = \int_{\Omega} -K_f u_3 U_3\, d\Omega + \int_{\Omega}\left(G_f \nabla^2 u_3\right)U_3\, d\Omega$$

$$= \int_{\Omega} -K_f u_3 U_3\, d\Omega + \int_{\Omega}\left(G_f \nabla^2 u_3\right)U_3\, d\Omega \tag{38}$$

The first integral on the right hand side will remains as it is, whereas the second integral can be decomposed using the integration by parts procedures (Green's second identity) twice (in similar manner as we used before for Laplace equation), to give:

$$\int_{\Omega}\left(G_f \nabla^2 u_3\right)U_3\, d\Omega = \int_{\Omega} G_f \left(\nabla^2 U_3\right)u_3\, d\Omega$$

$$+ \int_{\Gamma} G_f U_3 u_{3,n}\, d\Gamma - \int_{\Gamma} G_f U_{3,n} u_3\, d\Gamma \tag{39}$$

Then the final integral equation can be written as:

$$\int_{\Gamma}\left(-U_{ki}p_i + P_{ki}u_i\right)d\Gamma - \int_{\Gamma} G_f U_3 u_{3,n}\, d\Gamma + \int_{\Gamma} G_f U_3 u_{3,n}\, d\Gamma$$

$$-\int_{\Omega}\left(\mathbf{M}_{k\alpha\beta,\beta} - \mathbf{Q}_{k\alpha}\right)u_\alpha d\Omega + \int_{\Omega}\left(\mathbf{Q}_{k\alpha,\alpha} - k_f U_3 - G_f \nabla^2 U_3\right)u_3 d\Omega = 0 \tag{40}$$

And the fundamental solution cam be obtained from:

$$\mathbf{M}_{k\alpha\beta,\beta}(\xi,\mathbf{x}) - \mathbf{Q}_{k\alpha}(\xi,\mathbf{x}) = -\delta(\xi,\mathbf{x})\delta_{k\alpha} \tag{41}$$

$$\mathbf{Q}_{k\alpha,\alpha} - k_f U_3 - G_f \nabla^2 U_3 = -\delta(\xi,\mathbf{x})\delta_{k3} \tag{42}$$

Making use of the properties of the Dirac Delta, one can write:

$$u_i(\xi) + \int_{\Gamma(\mathbf{x})} P_{ki}(\xi,\mathbf{x})u_i(\mathbf{x})d\Gamma(\mathbf{x}) + G_f \int_{\Gamma(\mathbf{x})} U_{i3,n}(\xi,\mathbf{x})u_3(\mathbf{x})d\Gamma(\mathbf{x})$$

$$= \int_{\Gamma(\mathbf{x})} U_{ki}(\xi,\mathbf{x})p_i(\mathbf{x})d\Gamma(\mathbf{x}) + G_f \int_{\Gamma(\mathbf{x})} U_{i3}(\xi,\mathbf{x})u_{3,n}(\mathbf{x})d\Gamma(\mathbf{x}) \tag{43}$$

Which is the required boundary integral equation. For more details on the derivation, the reader can consult Ref. [5].

References and Further Reading

[1] Kythe, P.K., Fundamental Solutions for Differential Operators and Applications, Birkhäuser Press, Berlin, Germany, 1996.

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