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The birth of the boundary element method from conception to application

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ABSTRACT

The Boundary Element Method (BEM) has now become a well established numerical technique with a number of computer programmes to its credit oriented towards industrial applications. They are reliable and robust tools which stress its unique features versus finite elements, ie reduced dimensionality which makes it easier to interface it to CAD codes; better accuracy, the elegant way in which moving boundaries are dealt with and the possibility of taking into consideration infinite domains without the need of introducing artificial boundaries. An excellent paper on the early work that led to the development of boundary elements has been given by A & D Cheng [1] and from which many of the illustrations depicting famous scientists were taken. The present contribution aims to explain further how the methodology developed and consolidated towards the end of the 1970s, beginning of the 1980s.

The current state of the Method is the result of the work of a large number of researchers and software specialists. It has been a long and very successful undertaking since 1978 which is perceived to be the birth date of the technique, ie when the first conference on BEM was held at Southampton University. It was also then that the first book on the Method was published. The fundamentals and methodology, still those in use, were established at that moment.

To understand how the idea of the Boundary Element Method developed, one needs to refer to the state of the art in finite elements (FEM) in the 1970s. That method was based in terms of the minimisation of a functional such as the corresponding to minimum potential energy or similar. This was the basis of the so-called Rayleigh (1842–1919) and Ritz (1874–1909) technique (Fig. 1). Later on, the idea that functionals were not needed gave rise to the use of the Galerkin (1871–1945) Method, an idea that took some time to be accepted [1].

Although the history of Boundary Elements is largely independent of that, it uses similar shape functions as FEM for the distribution of the variables over the surface. Its basis however can be traced to integral equations theory.

The author's own introduction to integral equations came through the work of the Italian School, the influence of which was strong in latin countries. It included such outstanding scientists as Enrico Betti (1823–1892), renowned for his reciprocity principle and a source of inspiration for others such as Carlo Somigliana (1890–1955), whose identity for stress analysis can be directly applied to obtain BEM formulations and Vito Volterra (1860–1940) (Fig. 2). The last scientist originated a type of integral equation which could be used for time-dependent problems, including modelling of visco-elastic effects in concrete. Volterra's equations were applied in the author's first research laboratory (University of Rosario, Argentina) to study creep buckling of concrete columns and the results compared against a series of experiments. At that time, that work was carried out under Professor Nestor Distefano (1931–1975) late of the University of California at Berkeley, where he was to die suddenly at an early age, before he was able to develop his outstanding capabilities to their full potential. Nestor was exceptionally intelligent as well as possessing an inquisitive mind; always at his best when working in a multi-disciplinary environment.

It was after moving to the UK that the author started to appreciate the advantages of more general integral equation formulations and in particular the work of George Green (1793–1841) on setting up the basis for direct integral methods and Erik Fredholm (1866–1927), who contributed to the theory of indirect integral equations (Fig. 3).

BEM is nowadays mostly based on the direct formulation and hence owes much to the pioneering work of George Green. He published his first and most influential papers on a subscription basis; ie for a group of people prepared to buy each new article, something that could only have taken place in the British Enlightenment period, when society as a whole was closely following all major scientific developments. The fact is that few of his subscribers were able to follow Green's work and his ideas lay dormant until rediscovered by Lord Kelvin, who understood their importance.

It was also due to the influence of several other mathematical and engineering scientists that the BEM was born. The original fundamentals came from the Southampton University Group where the author was carrying out his research, inspired by his Supervisor, Professor Hugh Tottenham (1926–2012), a brilliant researcher who unfortunately left very few publications behind. He was the inspiration of the whole group working on engineering mechanics at the Department of Civil Engineering, and his influence was marked in the field of integral equations (Fig. 4).

Tottenham studied at Cambridge at a time when many intellectuals saw Russia and the communist regime in a favourable light. He learnt Russian during his studies and was able to convey to his disciples the work of many mathematicians working in that country on integral equations at a time when there were few translations available. This referred specially to the Georgian School, represented by Nikolai Muskhelishvili (1891–1974) and Viktor Kupradze (1903–1985), the latter one of the more important references on direct boundary integrals. Tottenham served until his death as a member of the Board of Directors of the Wessex Institute of Technology (WIT), where work on BEM was to continue until now.

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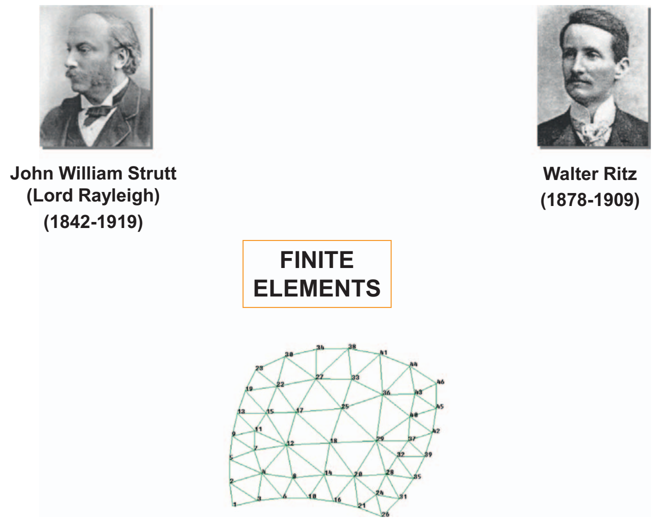


Fig. 1. The FEM was originally expressed in terms of Rayleigh Ritz Method, a technique based on the minimisation of a function.

Fig. 2. The Italian School of mechanical sciences, which led to Somigliana's identity and the Volterra integrals.

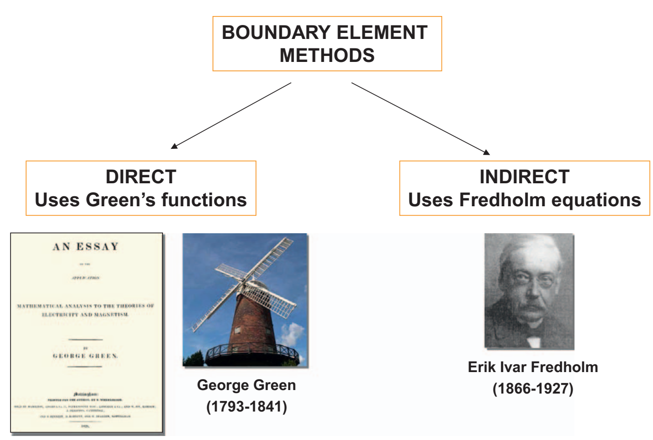


Fig. 3. The integral equation formulations originated the direct and indirect BEM.

Parallel to Tottenham's work, Prof Maurice Jaswon (1922–2011) of City University was also developing boundary integral formulations. His research gave origin to a book with George Symm (until a few years ago also a member of WIT's Board of Directors). The work of this Group somewhat petered out, possibly on the assumption that the contemporary emergence of FEM made any further research on other methods unnecessary. It was a mistake on their part.

An important by-product of Jaswon's work was the setting up of boundary integral equation research in the USA under Frank Rizzo (1938). This was the result of Maurice's time spent there when Rizzo realised the possibility of applying Somigliana's identity to obtain boundary integral formulations for stress analysis. A series of bright researchers were influenced by Rizzo, including Thomas Cruse (1941). Their excellent work has unfortunately not been continued.

At the end of 1960, beginning of 1970, the fundamentals of boundary integral equations and the parallel development of the FEM methodology were in place. The moment was right for the emergence of BEM but a further component was required.

This was provided by the work at MIT on mixed formulations; originally due to Eric Reissner (1913–1996) quasi variational principles which were extended by Jerry Connor (1932) from the Civil Engineering Department to finite elements. The author was fortunate enough to take a course with Reissner and even more to work with Jerry Connor, one of the most brilliant researchers the author has ever known. He is also a superb teacher

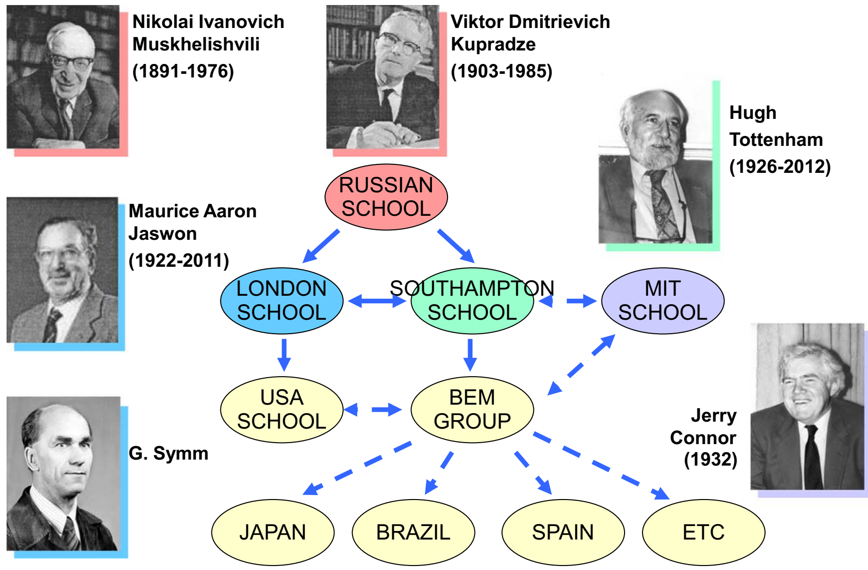
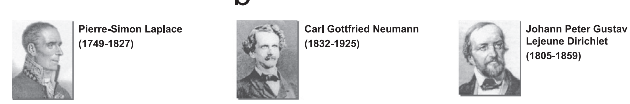


Fig. 4. The roots of BEM, based on the work of the Russian School, MIT and Southampton University, plus the contribution of Maurice Jaswon, who originated direct BEM in the USA.

a

Laplace equation

b



$$\nabla^2 u = 0 \quad in\ \Omega$$

$$u = f(x) \qquad x \in \Gamma_1$$

$$\frac{\partial u}{\partial n} = g(x) \qquad x \in \Gamma_2$$

Fig. 5. (a+b) The case of Laplace's equation as an example of how to apply BEM. a The Laplace equation and its domain of application. b The classical two types of boundary conditions for the Laplace equation, ie Neuman or essential and Dirichlet or natural.

a

w: weighting function

$$\int (\nabla^2 u) w\, d\Omega = \int_{\Gamma_2} (\frac{\partial u}{\partial n} - g) w\, d\Gamma - \int_{\Gamma_1} (u - f) \frac{\partial w}{\partial n} d\Gamma$$

b

IBP

$$\int u \nabla^2 w\, d\Omega = \int_{\Gamma_2} u \frac{\partial w}{\partial n} d\Gamma - \int_{\Gamma_1} \frac{\partial u}{\partial n} w\, d\Gamma + \int_{\Gamma_1} f \frac{\partial w}{\partial n} d\Gamma - \int_{\Gamma_2} g\, w\, d\Gamma$$

c

FUNDAMENTAL SOLUTION

$$u* = \frac{1}{4\pi r}$$

$$\nabla^2 u* = \delta$$

d

$$u = \int_{\Gamma_2} u \frac{\partial u*}{\partial n} d\Gamma - \int_{\Gamma_1} \frac{\partial u}{\partial n} u*\, d\Gamma + \int_{\Gamma_1} f \frac{\partial u*}{\partial n} d\Gamma - \int_{\Gamma_2} g\, u*\, d\Gamma$$

e

$$c^i u^i + \int_{\Gamma} u\, t*\, d\Gamma = \int_{\Gamma} t\, u* d\Gamma$$

$t* = \partial u*/\partial n$

t = αu/αn

f

$$c^i u^i + \int_{\Gamma} u\, t*\, d\Gamma = \int_{\Gamma} t\, u* d\Gamma$$

$\underline{H}\ \underline{U} = \underline{G}\ \underline{T}$

$\underline{AX} = \underline{B}$

Fig. 6. (a+f) The deduction of BEM System Matrices starting from mixed principles. a The weighted residual formulation of Laplace's equation and the two types of boundary conditions. b Integration by parts of the domain term to obtain the Laplacian of the weighting function. c Definition of the fundamental solution for three dimensional Laplace equation. d Introducing the fundamental solution reduced the domain integral to the value of the potential at a point. Valid for an internal point. e Taking the equation to the boundary at 'i' results in a coefficient c^i (equal to ½ for the smooth boundary). f Assuming element shapes on the boundary and Integrating produces the classical H and G matrix equations, which can be rearranged into A after application of boundary conditions.

and able to explain in simple terms the most convoluted themes. Jerry, who is also a Director of WIT, was at that time interested in the application of mixed principles in FEM.

Mixed principles are based on postulating the solution of a set of equations weighted by a series of Lagrangian multipliers. The choice of which equations to select and which others to satisfy identically are left to the user. The physical meaning of the Lagrangian Multipliers can be found by

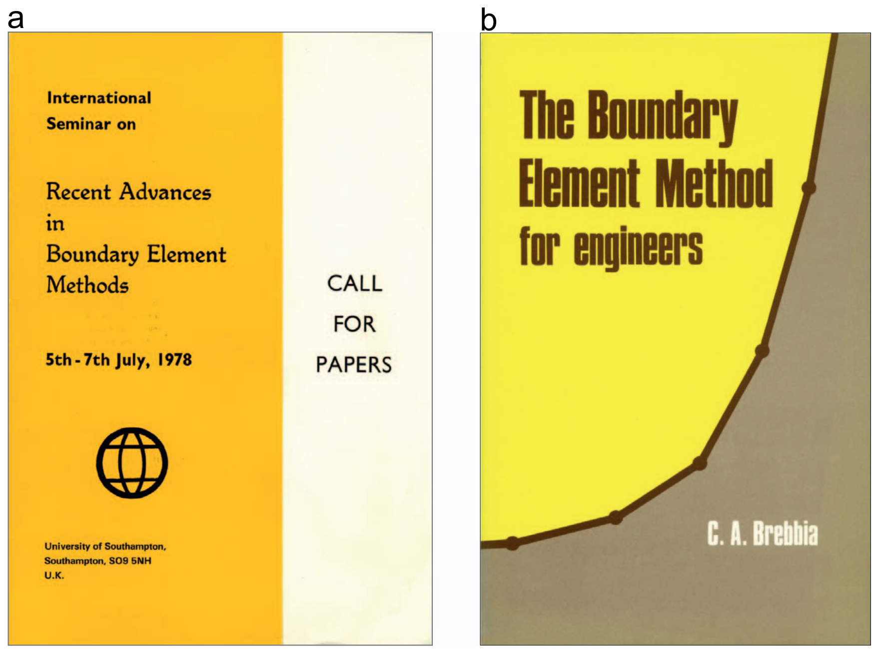


Fig. 7. The launching of the BEM. a Call for Papers from the 1st International Conference on BEM held at Southampton University in 1978. b Cover of the first book on the Method, published by Pentech Press and Wiley in 1978.

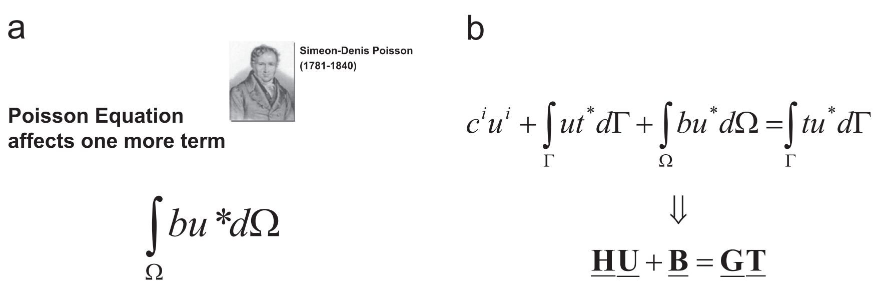


Fig. 8. The case of Poisson's equation, which contains a domain term. Notice now the appearance of a domain integral. a The domain integral corresponding to the terms on the right hand side of the Laplace equation. b Integral statement similar to the one in Fig. 6f but containing the new terms, which once integrated gives rise to an additional vector.

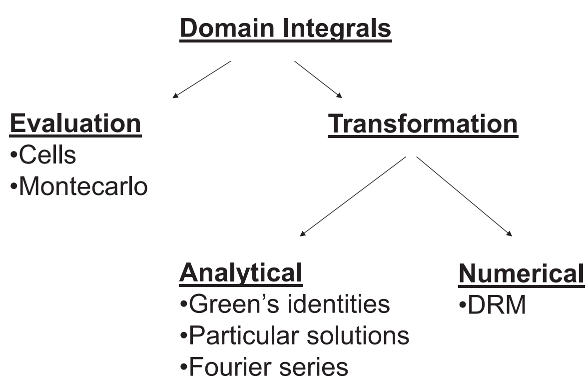


Fig. 9. Different ways in which the domain integral in cases like Poisson's equation can be evaluated in BEM. Direct evaluation can be carried out by dividing the domain into cells or integrating at random points as in Montecarlo techniques. Analytical techniques such as Green's identities or Fourier series analysis can also result in boundary-only problems, while the most frequently used technique is to find particular solutions in the cases where the domain term is known, ie a change of variables. The Dual Reciprocity Method (DRM) represents a novel way of applying particular solutions.

manipulation of the equations, using simple integration by parts. The formulation can be seen as a generalised expression of virtual work, a principle that can be dated to Aristotle.

From the numerical point of view, the Lagrangian multipliers can be interpreted as weighting functions which furthermore can be used to distribute the errors made by assuming certain approximate functions, such as polynomials for the variables.

The emergence of numerical formulations was essential to understand how different approximation methods could be obtained and relate to each other (refer to Classification in [3]).

The author was fortunate enough to be in the right place at the right time and in realising that using mixed principles, any resulting boundary errors could be distributed through weighting functions.

Particular Integrals

$$u = \tilde{u} + \bar{u}; \qquad t = \tilde{t} + \bar{t}$$

To satisfy b. term.

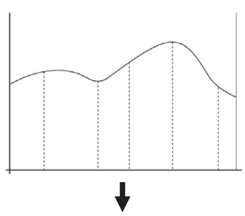
$$c^i u^i + \int_\Gamma u t^* d\Gamma - \int_\Gamma t u^* d\Gamma$$

$$- \left\{ c^i \bar{u}^i + \int_\Gamma \bar{u} t^* d\Gamma - \int_\Gamma \bar{t} u^* d\Gamma \right\} = 0$$

$$\underline{\mathbf{H}}\underline{\mathbf{U}} - \underline{\mathbf{G}}\underline{\mathbf{T}} = \left[\underline{\mathbf{H}}\underline{\bar{\mathbf{U}}} - \underline{\mathbf{G}}\underline{\bar{\mathbf{T}}} \right]$$

Fig. 10. Case of using a particular solution appropriate for the whole domain term. In this case the solution can be divided into that of the homogenous (Laplace) equation and the particular integral satisfying the b term.

a



$$b = \sum_j \alpha_j f_j$$

b

DUAL RECIPROCITY METHOD

$$b = \sum f\alpha \quad \Rightarrow \quad \begin{array}{l} \underline{\mathbf{b}} = \underline{\mathbf{F}}\underline{\boldsymbol{\alpha}} \\ \underline{\boldsymbol{\alpha}} = \underline{\mathbf{F}}^{-1}\underline{\mathbf{b}} \end{array}$$

$$\nabla^2 \bar{u}_j = f_j$$

$$\int_\Omega b u^* d\Omega = \sum \left\{ \int_\Omega \alpha_j \left(\nabla^2 \bar{u}_j \right) u^* d\Omega \right\}$$

c

RHS

$$\sum \left\{ \int_\Omega \alpha_j \left(\nabla^2 \bar{u}_j \right) u^* d\Omega \right\} =$$

$$= \sum \alpha_j \left\{ c_j \bar{u}_j + \int_\Gamma t^* \bar{u}_j d\Gamma - \int_\Gamma u^* \bar{t}_j d\Gamma \right\}$$

d

$$\sum \{ \underline{\mathbf{H}}\underline{\mathbf{U}} - \underline{\mathbf{G}}\underline{\mathbf{T}} \} \underline{\boldsymbol{\alpha}}$$

The final expression becomes:

$$\underline{\mathbf{H}}\underline{\mathbf{U}} - \underline{\mathbf{G}}\underline{\mathbf{T}} = \sum \{ \underline{\mathbf{H}}\underline{\bar{\mathbf{U}}} - \underline{\mathbf{G}}\underline{\bar{\mathbf{T}}} \} \underline{\boldsymbol{\alpha}}$$

e

RBF

$\left(r^2 + c^2\right)^{n - \frac{3}{2}}$	MULTIQUADRICS	
$r^{2n} \ln r$	in 2D	BREBBIA
r^{2n-1}	in 3D	
$\exp\left(-\frac{r^2}{c^2}\right)$	GAUSSIAN	

Fig. 11. The case of localised particular solutions, which gives origin to the Dual Reciprocity Method (DRM) for dealing with a general type of b term. a Description of the function b in terms of a series of parameters α and known localised function f, which are called radial basis functions. b Assuming that the function f is of the same type at each point of application, one can find a particular solution ū which will be generic. The function b on the whole domain can then be expressed in function of α's parameters and in localised particular solutions. c Simple integration by parts converts the domain term into boundary integrals only. d The matrix expression for the initial b function involves the same H and G matrices previously encountered in the left hand side solution (ie for Laplace's operator). This allows for considerable simplicity in the calculations. e Some of the Radial Basis Functions. Notice that some of them require the determination of extra parameters. Because of that it tends to give more reliable results using the simple logarithm and radial functions shown as Brebbia's.

The fundamentals of boundary elements will now be presented by way of a series of illustrations, taking as an example the solution of the well known Laplace Equation with relevant boundary conditions, as shown in Fig. 5.Contrary to usual FEM, in BEM one chooses the weighting functions as different from the ones for approximating the variables. This led to using 'fundamental solutions', ie those which satisfy the equation or set of equations, in the domain but without any reference to the boundary conditions. By doing that domain integrals vanish and the dimensionality of the problem is reduced by one (Fig. 6).

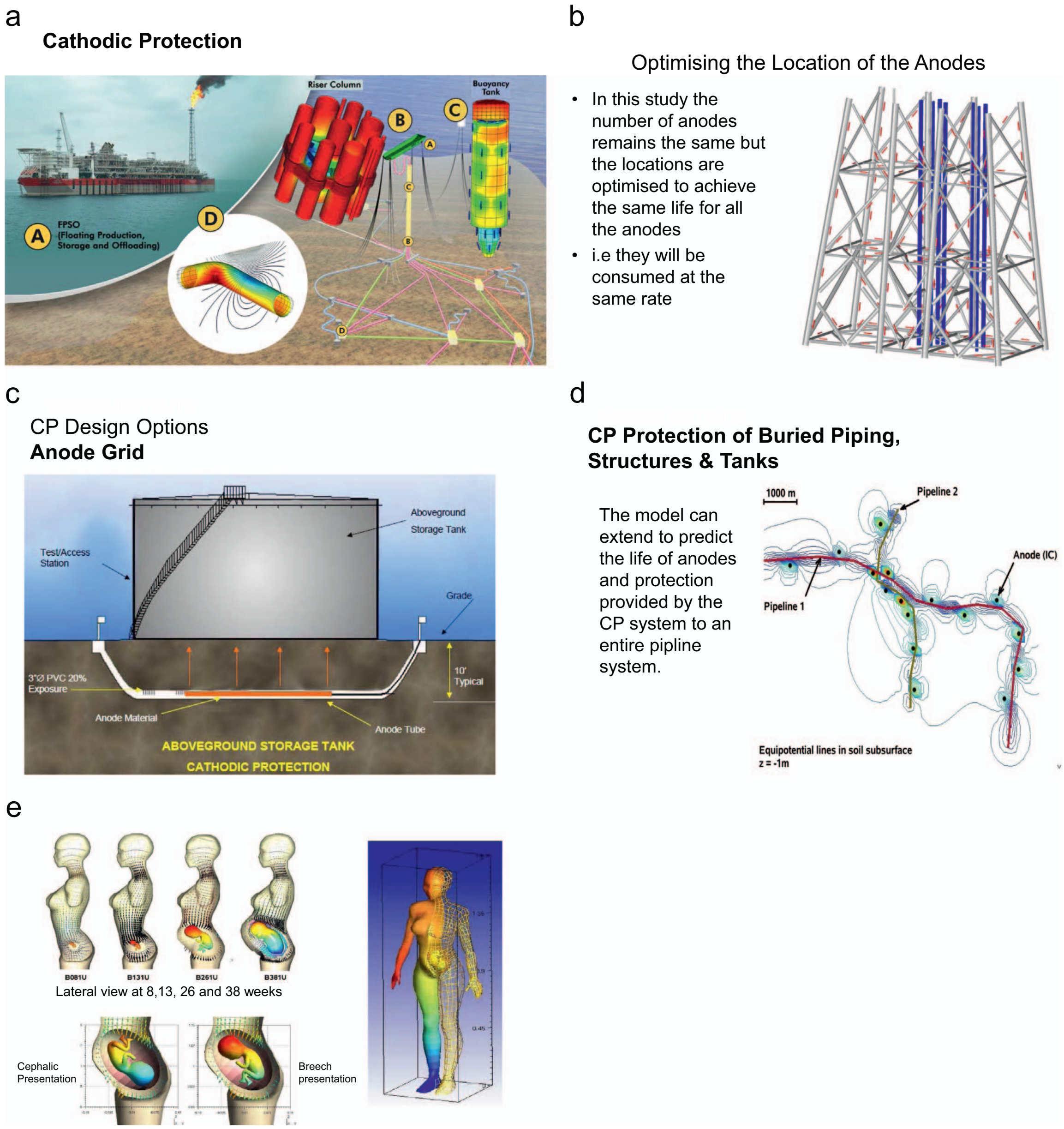


Fig. 12. A series of practical applications for which the domain extends to infinity and are hence ideally suited for BEM solution. a Examples related to the cathodic protection of offshore systems. The BEM solves an electric condition problem, with water as electrolyte and elements only required on the surface of the different components. b Another Cathodic Protection System where BEM is applied to optimise the position of the anodes to maximise the life of the structure. c Cathodic Protection System for the protection of onshore structures such as oil tanks are now routinely analysed using BEM. Here the mechanism extending to infinity is the soil rather than the water. d Pipelines require a different type of protection. Here packs of anodes are located along the pipeline to generate the electric field required for their protection. BEM can optimise their position. e Electromagnetic radiation can affect living tissues. The figure depicts a BEM representation of the human body which can include different organs. In this case, the study consisted in determining the level of radiation produced by ELF (Extremely Low Frequency) on the foetus.

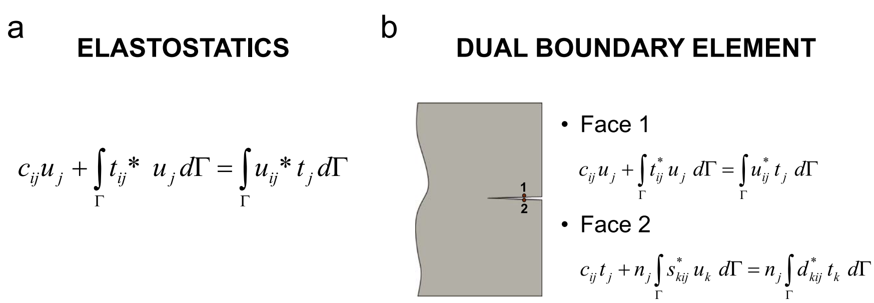


Fig. 13. The case of elastostatics problems and the development of the Dual Boundary Element Method for cracks. a The integral equation for an electrostatics problem with u and t being tensorial components corresponding to displacements and surface tractions. Their relationship, which can be obtained using weighted residuals, can also be found by applying Somigliana's principle. b The Dual Boundary Element Method consists of applying the integral expression in the previous Fig. 13a on one face of the crack and a high order on the other face (the latter is obtained by differentiating on the first).

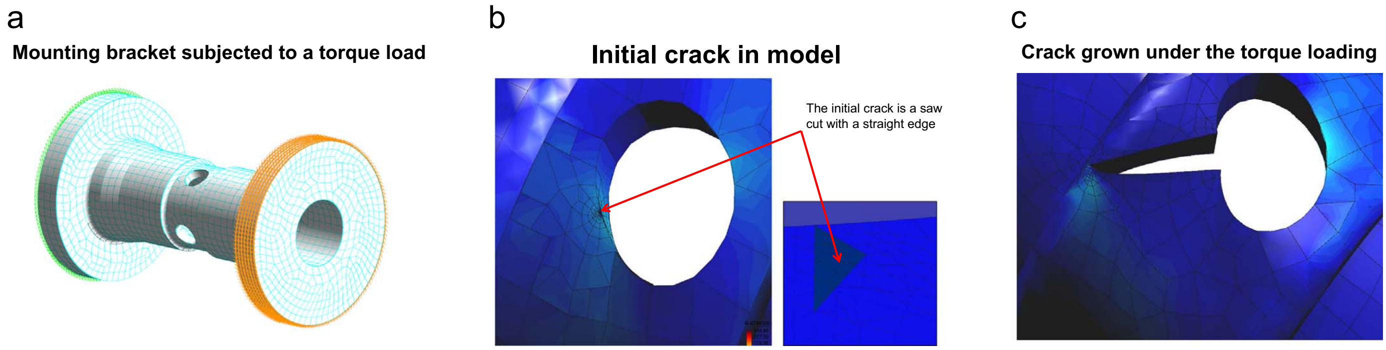


Fig. 14. Mounting bracket subject to a torque load. a Geometry of the bracket showing the Boundary element mesh. b An initial small “saw cut” crack has been added to the bracket on the inside surface of one of the circular holes. c Under torque loading the crack starts to grow. The Dual Boundary Element Method adds new elements to the growing surface of the crack without need to alter the rest of the mesh.

As is now well known, the functions used in the surface or boundary are similar to those polynomials employed in FEM while the unique properties of the fundamental solution improves the accuracy of the results.

The coming together of all these concepts led to the BEM in its present form being born, an idea which consolidated during the 1978 first BEM Conference held at Southampton University. The contemporary book “The Boundary Element Method for Engineers” [2] (Pentech Press, Wiley, 1978) presented the methodology and the notation still in use today (Fig. 7). This Volume was followed at a later stage by two more comprehensive works [3,4].

Some of the advantages of the BEM were fully appreciated then, including its reduced dimensionality, the possibility of solving problems extending to infinity without a domain mesh and the high accuracy of the results. There was no need to stress the obvious elegance and conceptual simplicity of the Method, but other important properties only became apparent as the research progressed.

Researchers started to realise the advantages of having a reduced order of continuity as a result of using different types of independent variables, say displacements and surface stresses. This led to the simple treatment of discontinuities and singularities and simplicity in meshing.

This also produced further work on moving boundaries, for cases such as Stefan's problems, bubble evolution, wave analysis, and many others.

What took some time to be understood is that the new weighted residual type BEM formulations easily extended the range of problems to be solved. BEM could now attempt to solve non-linear as well as time-dependent problems. One could produce weighted residual statements in which any type of term could be included. The formulation produced however domain integrals requiring some form of domain integration (Fig. 8).

This led to an active research period at the Southampton University Group, during which many solutions were proposed to transfer domain integrals to the boundary using analytical or approximate techniques (Fig. 9).

It was then that a new approach to the problem was proposed. It was based on the idea that to solve a non-homogenous differential equations one can propose separation of variables such that they can be expressed as a combination of homogenous and particular solution.

Particular solutions however are based on the knowledge of the domain terms (Fig. 10). This in itself is not of much use if those terms are nonlinear or time-dependent.

At that moment the novel idea of proposing the use of an approximation for those domain terms was born. The idea, albeit highly original, was extremely simple, ie propose the domain term as composed of the summation of a series of localised functions, multiplied by a co-efficient which can be nonlinear or time-dependent (Fig. 11).

The key of the technique was proposing simple functions which give rise to equally simple particular solutions. The Method basically a substitution of variables does not require the calculation of new matrix co-efficients, an operation that can be time-consuming due to the singular character of the fundamental solution. The simplicity of the Dual Reciprocity Method [5,6] as the technique was called, led to its wide application.

The next step in the development of BEM was its computer implementation to solve problems in engineering practice and this work accelerated when the Southampton University group moved to the newly founded Wessex Institute of Technology (WIT) in 1986. The Institute, which is now located on its own Campus in the New Forest, UK, continues to work on extensions of the Method to solve a wide diversity of problems as well as carries out research into the optimisation of its computer performance.

Boundary Elements range of applications continues to increase steadily. Its unique properties regarding representation of problems extending to infinity, makes it an ideal solution for problems such as structures in the sea, foundations and pipelines on-shore, wave propagation in general, including electromagnetics (Fig. 12).

Many boundary problems have, for a long time, been solved by BEM in preference to other methods. It was for instance in 1992 when work at WIT started on how to solve crack propagation problems. This led to what is now called the Dual Boundary Element Method [7], the term “dual” referring to the possibility of defining two different equations for the opposite sides of a crack in order to avoid singular solutions. This was done by proposing a second equation based on operating on the original (such as computing the derivatives of displacements, obtaining from the combination of strains, stresses and finally surface forces). The two equations on the crack surfaces in the case of stress analysis are the displacements and surface stress boundary integral expressions (Fig. 13).

The Dual BEM allowed not only for the accurate computation of stress intensity factors but also for crack propagation studies; when adding more elements as the crack progresses (Fig. 14).

Further novel applications of BEM are those in biomedicine, water resources, fluid mechanics in general and many others.

BEM has become well established in engineering analysis and design and as time advances its unique properties will be more fully appreciated, leading as well to more frequent combination of the method with FEM. Both techniques can be seen as complimenting each other, rather than in competition.

Acknowledgment

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